Burning Random Trees

Luc Devroye[*] Austin Eide[†] Paweł Prałat[‡]

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Abstract

Let $\mathcal{T}$ be a Galton-Watson tree with a given offspring distribution ξ, where ξ is a $\mathbb{Z}_{\geq 0}$-valued random variable with $\mathsf{E}[\xi] = 1$ and $0 < \sigma^2 := \mathsf{Var}[\xi] < \infty$. For $n \geq 1$, let T_n be the tree $\mathcal{T}$ conditioned to have n vertices. In this paper we investigate $b(T_n)$, the burning number of T_n. Our main result shows that asymptotically almost surely $b(T_n)$ is of the order of $n^{1/3}$.

1 Introduction

Graph burning is a discrete-time process that models influence spreading in a network. Vertices are in one of two states: either *burning* or *unburned*. In each round, a burning vertex causes all of its neighbours to burn and a new *fire source* is chosen: a previously unburned vertex whose state is changed to burning. The updates repeat until all vertices are burning. The *burning number* of a graph G, denoted $b(G)$, is the minimum number of rounds required to burn all of the vertices of G.

Graph burning first appeared in print in a paper of Alon [2], motivated by a question of Brandenburg and Scott at Intel, and was formulated as a transmission problem involving a set of processors. It was then independently studied by Bonato, Janssen, and Roshanbin [5, 6] who, in addition to introducing the name *graph burning*, gave bounds and characterized the burning number for various graph classes. The problem has since received wide attention (e.g. [3, 4, 11, 14, 15, 16]), with particular focus the so-called Burning Number Conjecture that each connected graph on n vertices requires at most $\lceil \sqrt{n} \rceil$ turns to burn. This conjecture is best possible, as $b(P_n) = \lceil \sqrt{n} \rceil$ for a path on n vertices.

Clearly, $b(G) \leq b(T)$ for every spanning tree T of G. Hence, the Burning Number Conjecture can be stated as follows:

Conjecture 1.1. *For any tree T on n vertices, $b(T) \leq \lceil \sqrt{n} \rceil$.*

[*]School of Computer Science, McGill University, Montreal, Canada
[†]Department of Mathematics, Toronto Metropolitan University, Toronto, Canada
[‡]Department of Mathematics, Toronto Metropolitan University, Toronto, Canada

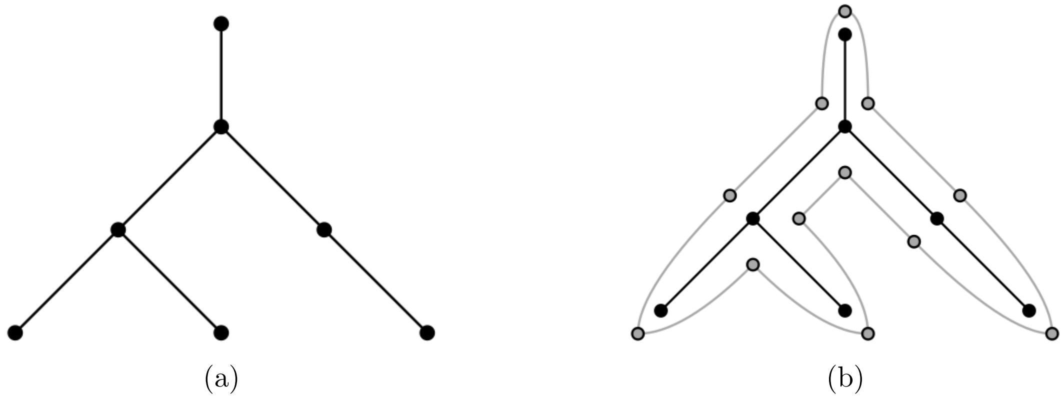

Figure 1: A tree T (a) and a cycle corresponding to a depth-first search of T (b). Any burning sequence for the cycle projects to a burning sequence for the tree.

Although the conjecture feels obvious, it has resisted attempts at its resolution. It is easy to couple the process on any tree T on n vertices ($n-1$ edges) with the process on $C_{2(n-1)}$, a cycle on twice as many edges. (See Figure 1.) This yields

$$b(T) \le \left\lceil \sqrt{2(n-1)} \right\rceil = \sqrt{2n} + O(1) \qquad (\sqrt{2} \approx 1.41421).$$

In [4], it was proved that

$$b(T) \le \sqrt{\frac{12}{7}n} + 3 \qquad (\sqrt{12/7} \approx 1.30931).$$

This bound was consecutively improved in [11] to

$$b(T) \le \left\lceil \frac{\sqrt{24n+33}-3}{4} \right\rceil = \sqrt{\frac{3}{2}n} + O(1) \qquad (\sqrt{3/2} \approx 1.22475),$$

and currently, the best upper bound proved in [3], is

$$b(T) \le \left\lceil \sqrt{\frac{4}{3}n} \right\rceil + 1 \qquad (\sqrt{4/3} \approx 1.15470).$$

Arguably, the strongest result in this direction shows that the conjecture holds asymptotically [16], that is,

$$b(T) \le (1+o(1))\sqrt{n}.$$

We intend to investigate the burning number of random trees. Let $\mathcal{T}$ be a Galton-Watson tree with a given offspring distribution ξ, where ξ is a $\mathbb{Z}_{\ge 0}$-valued random variable with

$$\mathsf{E}[\xi] = 1, \qquad \text{and} \qquad 0 < \sigma^2 := \mathsf{Var}[\xi] < \infty. \tag{1}$$

In other words, the Galton-Watson tree is critical, of finite variance, and satisfies $\mathbb{P}(\xi = 1) < 1$. In particular, it implies that $0 < \mathbb{P}(\xi = 0) < 1$.

For $n \geq 1$, let T_n be the tree $\mathcal{T}$ conditioned to have n vertices. The resulting random trees are essentially the same as the simply generated families of trees introduced by Meir and Moon [13]. This family contains many combinatorially interesting random trees such as uniformly chosen random plane trees, random unordered labelled trees (known as Cayley trees), and random d-ary trees. For more examples, see, Aldous [1] and Devroye [7].

Our main result shows that, with high probability, $b(T_n)$ is of the order of $n^{1/3}$.

Theorem 1.2. *Let T_n be a conditioned Galton-Watson tree of order n, subject to (1). For any $\epsilon = \epsilon(n)$ tending to 0 as $n \to \infty$, we have*

$$\mathsf{Pr}\Big((\epsilon n)^{1/3} \leq b(T_n) \leq (n/\epsilon)^{1/3}\Big) = 1 - O(\epsilon).$$

The paper is structured as follows. First, we make a simple observation that the burning number can be reduced to the problem of covering vertices of the graph with balls, a slightly easier problem; see Section 2. Section 3 is devoted to the lower bound for the burning number whereas the upper bound is provided in Section 4. We finish the paper with a few natural questions; see Section 5.

2 Covering a Graph with Balls

In this section, we show a simple but convenient observation that reduces the burning number to the problem of covering the graph's vertices with balls. Let $G = (V, E)$ be any graph. For any $r \in \mathbb{N}_0$ and vertex $v \in V$, we denote by $B_r(v)$ the ball of radius r centered at v, that is, $B_r(v) = \{u \in V : d(u, v) \leq r\}$, where $d(u, v)$ denotes the distance between u and v.

First, note that since the burning process is deterministic, a fire source v makes all vertices in $B_t(v)$ burn after t rounds but only those vertices are affected. As a result, the burning number can be reformulated as follows:

$$b(G) = \min\left\{k \in \mathbb{N} : \exists v_0, v_1, \ldots, v_{k-1} \in V \text{ such that } \bigcup_{r=0}^{k-1} B_r(v_r) = V\right\}.$$

Dealing with balls of different radii is inconvenient so we will simplify the problem slightly by considering balls of the same radii. Let $\hat{b}(G)$ be the counterpart of $b(G)$ for this auxiliary covering problem, that is,

$$\hat{b}(G) = \min\left\{k \in \mathbb{N} : \exists v_1, v_2, \ldots, v_k \in V \text{ such that } \bigcup_{r=1}^{k} B_k(v_r) = V\right\}.$$

Covering with k balls of increasing radii (in particular, all of them of radii at most $k-1$) is not easier than covering with k balls of radius k. Hence, $\hat{b}(G) \leq b(G)$. On the other hand, covering with $2k$ balls of increasing radii (in particular, k of them of radii at least k) is not more difficult than covering with k balls of radius k implying that $b(G) \leq 2\hat{b}(G)$. We conclude that $\hat{b}(G)$ and $b(G)$ are of the same order:

Observation 2.1. *For any graph $G = (V, E)$,*

$$\hat{b}(G) \leq b(G) \leq 2\hat{b}(G).$$

In particular, we may prove the bounds in our main result (Theorem 1.2) for $\hat{b}(G)$ instead of $b(G)$, which will be slightly easier.

3 Lower bound

For an arbitrary tree τ and $i \in \mathbb{N}$, let

$$P_i(\tau) := \left| \Big\{ \{v, w\} : v, w \in V(\tau),\, d(v, w) = i \Big\} \right|.$$

In other words, $P_i(\tau)$ counts the number of unordered pairs of vertices which are distance i apart in τ. We have the following result from [8] that upper bounds the expected number of such pairs in the random tree T_n:

Theorem 3.1 ([8, Theorem 1.3])**.** *There exists a constant $c > 0$, dependent on the distribution of ξ, such that for all $n \in \mathbb{N}$ and $i \in \mathbb{N}$, $\mathsf{E}[P_i(T_n)] \leq cni$.*

We use Theorem 3.1 to prove the following:

Proposition 3.2. *Let $k = k(n) = (\epsilon n)^{1/3}$, where $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$. With probability $1 - O(\epsilon)$, $\hat{b}(T_n) \geq k$, that is, there is no partition of the vertices of T_n into k disjoint sets $U_1, U_2, \ldots, U_k$ such that* $\operatorname{diam}(U_j) \leq 2k$ for all $1 \leq j \leq k$.

Since $b(T_n) \geq \hat{b}(T_n)$ (Observation 2.1), Proposition 3.2 implies the corresponding lower bound in Theorem 1.2.

Proof. Let $Q_j(T_n) = \sum_{i=1}^{j} P_i(T_n)$, that is, $Q_j(T_n)$ counts pairs of vertices in T_n which are at most distance j apart. From Theorem 3.1, for all $n, j \in \mathbb{N}$,

$$\mathsf{E}[Q_j(T_n)] = \sum_{i=1}^{j} \mathsf{E}[P_i(T_n)] \leq cn \sum_{i=1}^{j} i \leq cnj^2.$$

For a contradiction, suppose there is a partition $U_1, U_2, \ldots, U_k$ of the vertices of T_n as described in the statement of the proposition. Since every pair of vertices in a given U_j is at most distance $2k$ apart, we must have

$$Q_{2k}(T_n) \geq \sum_{j=1}^{k} \binom{|U_j|}{2} = \sum_{j=1}^{k} \left(\frac{|U_j|^2}{2} - \frac{|U_j|}{2} \right) = \frac{1}{2} \sum_{j=1}^{k} |U_j|^2 - \frac{n}{2}.$$

By Jensen's inequality, we get

$$\frac{1}{k} \sum_{j=1}^{k} |U_j|^2 \geq \left(\frac{1}{k} \sum_{j=1}^{k} |U_j| \right)^2 = \left(\frac{n}{k} \right)^2$$

and therefore

$$Q_{2k}(T_n) \geq \frac{n^2}{2k} - \frac{n}{2} = \frac{n^2}{2k}\left(1 - O(k/n)\right) = \frac{n^2}{2k}\left(1 - o(n^{-2/3})\right).$$

On the other hand, $\mathsf{E}[Q_{2k}(T_n)] \leq 4cnk^2$. Thus, by Markov's inequality,

$$\begin{aligned} \mathsf{Pr}\left(Q_{2k}(T_n) \geq \frac{n^2}{2k} - \frac{n}{2}\right) &\leq \frac{\mathsf{E}[Q_{2k}(T_n)]}{\frac{n^2}{2k}}\left(1 + o(n^{-2/3})\right) \\ &= O\left(\frac{k^3}{n}\right) = O(\epsilon). \end{aligned}$$

It follows that a partition of $V(T_n)$ with the stated properties exists with probability $O(\epsilon)$, which finishes the proof of the proposition. $\square$

4 Upper bound

For any rooted tree τ with root r, for $i \in \mathbb{N}_0$ we write $\ell_i(\tau) := \{v \in V(\tau) : d_\tau(r, v) = i\}$ for the set of vertices at depth i. Let $h(\tau) \in \mathbb{N}_0 \cup \{\infty\}$ be the height of τ, that is, $h(\tau) := \sup\{i : \ell_i(\tau) \neq \emptyset\}$. For any $v \in V(\tau)$, let τ_v the (full) sub-tree of τ rooted at v. For $k \in \mathbb{N}$, and $j \in \{0, 1, \ldots, k-1\}$, let

$$\mathcal{C}_k^j(\tau) := \bigcup_{i=0}^{\infty} \Big\{v \in \ell_{ik+j}(\tau) \,:\, h(\tau_v) \geq k\Big\}.$$

So $\mathcal{C}_k^j(\tau)$ consists of all vertices whose depth is j modulo k with subtrees of height at least k.

We first show that placing balls of radius $2k$ at the root and at each vertex in $\mathcal{C}_k^j(\tau)$ covers the vertices of τ.

Lemma 4.1. *Let τ be a tree rooted at r. For any $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, k-1\}$ we have*

$$V(\tau) = \bigcup_{v \in \mathcal{C}_k^j(\tau) \cup \{r\}} B_{2k}(v).$$

Proof. Fix $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, k-1\}$. Let $v \in V(\tau)$ and let i be the smallest non-negative integer such that $d(r, v) < ik+j$ and let a be the unique ancestor of v in $\ell_{(i-2)k+j}(\tau)$, or $a = r$ if $i \in \{0, 1\}$. Then a is either the root r, or $d(r, a) \equiv j \pmod{k}$ and $h(\tau_a) \geq k$. In either case, $a \in \mathcal{C}_k^j(\tau) \cup \{r\}$. Since $d(a, v) \leq 2k$, we have $v \in B_{2k}(a)$. This finishes the proof of the lemma. $\square$

Lemma 4.1 provides a scheme to cover a general rooted tree τ with balls of radius $2k$. Observe that for any j, $|\mathcal{C}_k^j(\tau)| \leq 2k - 1$ implies that $\hat{b}(\tau) \leq 2k$, and hence $b(\tau) \leq 4k$ by Observation 2.1. In particular, we conclude the following:

$$\text{if } \min_j |\mathcal{C}_k^j(\tau)| \leq 2k-1, \text{ then } b(\tau) \leq 4k. \tag{2}$$

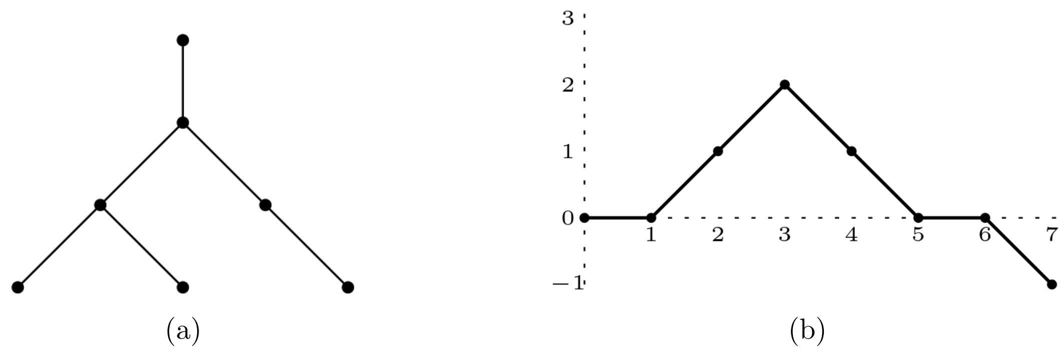

Figure 2: A rooted tree τ with preorder degree sequence $(1, 2, 2, 0, 0, 1, 0)$ (a) and its lattice path representation (b).

Our next lemma estimates the probability that a vertex selected uniformly at random from the random tree T_n has height at least k.

Lemma 4.2. *Consider the random tree $\tau = T_n$ on n vertices. Then there exists a constant $c > 0$, dependent on the distribution of ξ, such that the following property holds. Let u be a vertex selected uniformly at random from $V(\tau)$, and let its subtree be denoted by τ_u. Then, for all $k \geq 0$,*

$$\mathsf{Pr}\Big(h(\tau_u) \geq k\Big) \leq c\left(\frac{1}{k} + \frac{1}{\sqrt{n}}\right).$$

To prove Lemma 4.2 we require some standard tools for conditioned Galton Watson trees, which we introduce now. Given a tree τ with s vertices, let $v_1, v_2, \ldots, v_s$ be the vertices of τ in depth-first search (DFS) order. We write d_i for the number of children of v_i and refer to $(d_1, d_2, \ldots, d_s)$ as the *preorder degree sequence* of τ. The preorder degree sequence gives rise to a representation of τ as a lattice path $(j, y_j)_{j=0}^{s}$ started from $(0, y_0) = (0, 0)$ with s steps, where the jth step is given by $y_j = y_{j-1} + (d_j - 1)$. See Figure 2 for an example.

Note that the lattice path corresponding to a tree with s vertices always ends at the point $(s, y_s) = (s, -1)$ and has a height strictly greater than -1 before then, evidenced by the fact that the height of the path at step j is one less than the number of vertices in the "queue" of the DFS of the tree after j vertices have been explored. As the next lemma summarizes, there is a bijective correspondence between ordered trees with s vertices and lattice paths of this type.

Lemma 4.3 ([9, Lemma 15.2])**.** *A sequence $(d_1, d_2, \ldots, d_s) \in \mathbb{N}_0^s$ is the preorder degree sequence of a tree if and only if*

$$\sum_{i=1}^{j} d_i \geq j \quad \forall\, j \in \{1, 2, \ldots, s-1\}$$

and

$$\sum_{i=1}^{s} d_j = s - 1.$$

We also have the following useful property.

Lemma 4.4 ([9, Corollary 15.4])**.** *If $(d_1, d_2, \dots, d_s) \in \mathbb{N}_0^s$ satisfies $\sum_{i=1}^s d_i = s-1$, then precisely one of the cyclic permutations of $(d_1, d_2, \dots, d_s)$ is the preorder degree sequence of a tree.*

Let $(\xi_1^{(s)}, \xi_2^{(s)}, \dots, \xi_s^{(s)})$ be the preorder degree sequence of a Galton-Watson tree T_s with offspring distribution ξ conditioned to have s vertices, and let $(\widetilde{\xi}_1^{(s)}, \widetilde{\xi}_2^{(s)}, \dots, \widetilde{\xi}_s^{(s)})$ be a uniformly random cyclic permutation of $(\xi_1^{(s)}, \xi_2^{(s)}, \dots, \xi_s^{(s)})$. Let $\xi_1, \xi_2, \dots$ be a sequence of i.i.d. copies of ξ, and define, for any $j \ge 1$, $S_j = \sum_{i=1}^j \xi_i$. Lemmas 4.3 and 4.4 yield the following corollary:

Corollary 4.5. *The sequence $(\widetilde{\xi}_1^{(s)}, \widetilde{\xi}_2^{(s)}, \dots, \widetilde{\xi}_s^{(s)})$ has the same distribution as the sequence $(\xi_1, \xi_2, \dots, \xi_s)$ conditioned on $S_s = s-1$.*

Define the *span* of ξ as

$$h = \gcd\{i \ge 1 \,:\, \mathsf{Pr}(\xi = i) > 0\}.$$

We will use the following local limit theorems (see [10, Lemma 4.1 and (4.3)] and the sources referenced therein).

Lemma 4.6. *Suppose ξ satisfies (1) and has span h. Then, as $s \to \infty$, uniformly for $m \equiv 0 \pmod h$,*

$$\mathsf{Pr}(S_s = m) = \frac{h}{\sqrt{2\pi\sigma^2 s}} \left(e^{-(m-s)^2/2s\sigma^2} + o(1) \right).$$

If $\mathcal{T}$ is the Galton-Watson tree with offspring distribution ξ, then for $s \equiv 1 \pmod h$, as $s \to \infty$,

$$\mathsf{Pr}(|\mathcal{T}| = s) = \frac{h}{\sqrt{2\pi\sigma^2 s^3}}(1 + o(1)).$$

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. Throughout the proof, we will use $c_1, c_2, \dots$ for non-explicit positive constants which do not depend on n (but may depend on the distribution of ξ). Implicitly, we will assume throughout that $n \equiv 1 \pmod h$, where h is the span of ξ, so that $\mathsf{Pr}(S_n = n-1) > 0$.

We identify the random vertex u with a random index of the DFS order on T_n. Consider $(\xi_u^{(n)}, \xi_{u+1}^{(n)}, \dots, \xi_n^{(n)}, \xi_1^{(n)}, \dots, \xi_{u-1}^{(n)})$. It is clear that this sequence has the same distribution as $(\widetilde{\xi}_1^{(n)}, \widetilde{\xi}_2^{(n)}, \dots, \widetilde{\xi}_n^{(n)})$, which, in turn, has the same distribution as $(\xi_1, \xi_2, \dots, \xi_n)$ conditioned on $S_n = n-1$ by Corollary 4.5.

For $k \in \mathbb{N}$ and $s \ge k$, let $\mathfrak{T}_s^k$ be the set of ordered trees with s vertices and height at least k. For a sequence $(d_1, d_2, \dots, d_s) \in \mathbb{N}_0^s$, we write $(d_1, d_2, \dots, d_s) \in \mathfrak{T}_s^k$ if $(d_1, d_2, \dots, d_s)$

is the preorder degree sequence of a tree in $\mathfrak{T}_s^k$. Note that for any $(d_1, d_2, \ldots, d_s) \in \mathfrak{T}_s^k$, we have $\sum_{i=1}^{s} d_i = s - 1$. Then,

$$\begin{aligned}
\Pr(h(\tau_u) \geq k) &= \sum_{s=k}^{n} \Pr((\widetilde{\xi}_1^{(n)}, \widetilde{\xi}_2^{(n)}, \ldots, \widetilde{\xi}_s^{(n)}) \in \mathfrak{T}_s^k) \\
&= \sum_{s=k}^{n} \Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k \mid S_n = n-1) \\
&= \sum_{s=k}^{n} \frac{\Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k,\ S_n = n-1)}{\Pr(S_n = n-1)} \\
&= \sum_{s=k}^{n} \frac{\Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k) \cdot \Pr\left(\sum_{i=s+1}^{n} \xi_i = n-s\right)}{\Pr(S_n = n-1)} \\
&= \sum_{s=k}^{n} \frac{\Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k) \cdot \Pr(S_{n-s} = n-s)}{\Pr(S_n = n-1)}.
\end{aligned} \tag{3}$$

By Lemma 4.6, there is a constant $c_1 > 0$ such that for $s \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, we have

$$\frac{\Pr(S_{n-s} = n-s)}{\Pr(S_n = n-1)} \leq c_1.$$

Thus,

$$\begin{aligned}
\sum_{s=k}^{\lfloor n/2 \rfloor} \frac{\Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k) \cdot \Pr(S_{n-s} = n-s)}{\Pr(S_n = n-1)} &\leq c_1 \sum_{s=k}^{\lfloor n/2 \rfloor} \Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k) \\
&\leq c_1 \sum_{s=k}^{\infty} \Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k) \\
&= c_1 \Pr(h(\mathcal{T}) \geq k),
\end{aligned}$$

where, recall, $\mathcal{T}$ is the unconditioned Galton-Watson tree with offspring distribution ξ. By Kolmogorov's Theorem [12, Theorem 12.7], there is a constant $c_2 > 0$ so that for any $k \geq 1$, we have $\Pr(h(\mathcal{T}) \geq k) \leq c_2/k$, and thus $c_1 \Pr(h(\mathcal{T}) \geq k) \leq c_3/k$ for some constant $c_3 > 0$. This bounds the partial sum of (3) where $k \leq s \leq n/2$.

For the other partial sum, we first observe that

$$\Pr((\xi_1, \xi_2, \ldots, \xi_s) \in \mathfrak{T}_s^k) \leq \Pr(|\mathcal{T}| = s).$$

By Lemma 4.6, there is a constant $c_4 > 0$ so that

$$\frac{\Pr(|\mathcal{T}| = s)}{\Pr(S_n = n-1)} \leq \frac{c_4}{n}$$

for all $n \geq 1$ and $\lceil n/2 \rceil \leq s \leq n$. Using Lemma 4.6 again, we get that for all $j \geq 1$

$$\Pr(S_j = j) \leq \frac{c_5}{\sqrt{j}}$$

for some constant $c_5 > 0$. It follows that

$$\begin{aligned}\sum_{s=\lceil n/2\rceil}^{n} \mathsf{Pr}(S_{n-s} = n - s) &= \sum_{j=0}^{n-\lceil n/2\rceil} \mathsf{Pr}(S_j = j) \\ &\leq 1 + c_5 \sum_{j=1}^{n-\lceil n/2\rceil} \frac{1}{\sqrt{j}} \\ &\leq c_6\sqrt{n}\end{aligned}$$

for some constant $c_6 > 0$, where the bound in the last line follows from a straightforward comparison of the sum with an integral. In all, we get that the partial sum of (3) with $n/2 \leq s \leq n$ is at most $c_7/\sqrt{n}$ (for some constant $c_7 > 0$) for all n.

Finally, combining the two bounds, we conclude that (3) is upper bounded by $c_8\left(\frac{1}{k} + \frac{1}{\sqrt{n}}\right)$ for some constant $c_8 > 0$, as desired. This finishes the proof of the lemma. □

We now have all the ingredients to finalize the upper bound. The next theorem, as discussed earlier (see (2)), implies that $b(\tau) \leq 4k = O((n/\epsilon)^{1/3})$ with probability $1 - O(\epsilon)$.

Theorem 4.7. *Let $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$, and let $k = \lfloor\left(\frac{n}{\epsilon}\right)^{1/3}\rfloor$. Then, with probability $1 - O(\epsilon)$, we have*

$$\min_{j\in\{0,1,\ldots,k-1\}} |\mathcal{C}_k^j(T_n)| \leq 2k - 1.$$

Proof. Let $X_j = |\mathcal{C}_k^j(T_n)|$ for $j \in \{0, 1, \ldots, k-1\}$, and let Y be the number of vertices v in $\tau = T_n$ such that $h(\tau_v) \geq k$. Clearly, we have the identity $Y = \sum_{j=0}^{k-1} X_i$.

Now, observe that

$$\min_{j\in\{0,1,\ldots,k-1\}} X_j \leq \frac{1}{k}\sum_{j=0}^{k-1} X_j = \frac{Y}{k}.$$

By Lemma 4.2, $\mathsf{E}[Y] \leq c\left(\frac{n}{k} + \sqrt{n}\right)$. Therefore, by Markov's inequality,

$$\mathsf{Pr}\left(\frac{Y}{k} > 2k - 1\right) \leq \frac{\mathsf{E}[Y]}{(2k-1)k} \leq \frac{c}{2k-1}\left(\frac{n}{k^2} + \frac{\sqrt{n}}{k}\right) = O\left(\frac{n}{k^3}\right) = O(\epsilon).$$

This completes the proof of the theorem. □

5 Future Directions

In this paper we showed that asymptotically almost surely (a.a.s.) $b(T_n)$ is close to $n^{1/3}$, that is, a.a.s. $n^{1/3}/\omega \leq b(T_n) \leq n^{1/3}\omega$, provided that $\omega = \omega(n) \to \infty$ as $n \to \infty$. Is it true that a.a.s. $b(T_n) = \Theta(n^{1/3})$, that is, a.a.s. $c_1 n^{1/3} \leq b(T_n) \leq c_2 n^{1/3}$ for some constants $c_2 > c_1 > 0$? It is possible that there exists a constant $c_3 > 0$ (possibly depending on σ^2) such that a.a.s. $b(T_n) = c_3 n^{1/3}(1 + o(1))$.

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