NOTE ON THE MULTICOLOUR SIZE-RAMSEY NUMBER FOR PATHS

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ABSTRACT. The size-Ramsey number $\hat{R}(F,r)$ of a graph F is the smallest integer m such that there exists a graph G on m edges with the property that any colouring of the edges of G with r colours yields a monochromatic copy of F. In this short note, we give an alternative proof of the recent result of Krivelevich that $\hat{R}(P_n,r) = O((\log r)r^2 n)$. This upper bound is nearly optimal, since it is also known that $\hat{R}(P_n,r) = \Omega(r^2 n)$.

1. Introduction

Following standard notation, we write $G \to (F)_r$ if any r-edge-colouring of G (that is, any colouring of the edges of G with r colours) yields a monochromatic copy of F. We define the *size-Ramsey number* of F as $\hat{R}(F,r) = \min\{|E(G)| : G \to (F)_r\}$; that is, $\hat{R}(F,r)$ is the smallest integer m such that there exists a graph G on m edges such that $G \to (F)_r$. For two colours (that is, for $r = 2$) the size-Ramsey number was first studied by Erdős, Faudree, Rousseau and Schelp [9].

In this note, we are concerned with the size-Ramsey number of the path P_n on n vertices. It is obvious that $\hat{R}(P_n,2) = \Omega(n)$ and it is easy to see that $\hat{R}(P_n,2) = O(n^2)$; for example, $K_{2n} \to (P_n)_2$. The exact behaviour of $\hat{R}(P_n,2)$ was not known for a long time. In fact, Erdős [8] offered \$100 for a proof or disproof that $\hat{R}(P_n,2)/n \to \infty$ and $\hat{R}(P_n,2)/n^2 \to 0$. This problem was solved by Beck [1] in 1983 who, quite surprisingly, showed that $\hat{R}(P_n,2) < 900n$. (Each time we refer to inequality such as this one, we mean that the inequality holds for sufficiently large n.) A variant of his proof, provided by Bollobás [5], gives $\hat{R}(P_n,2) < 720n$. Recently, the authors of this paper [6] used a different and more elementary argument that shows that $\hat{R}(P_n,2) < 137n$. The argument was subsequently tuned by Letzter [12] who showed that $\hat{R}(P_n,2) < 91n$, and then further refined by the authors of this paper [7] who showed that $\hat{R}(P_n,2) \le 74n$. On the other hand, the first nontrivial lower bound was provided by Beck [2] and his result was subsequently improved by Bollobás [4] who showed that $\hat{R}(P_n,2) \ge (1+\sqrt{2})n - O(1)$. The strongest known lower bound, $\hat{R}(P_n,2) \ge 5n/2 - O(1)$, was proved in [7].

Let us now move to the multicolour version of this graph parameter. It was proved in [7] that $\frac{(r+3)r}{4}n - O(r^2) \le \hat{R}(P_n,r) \le 33r4^r n$. It follows that $\hat{R}(P_n,r)$ is linear for any fixed value of r but the two bounds are quite apart from each other in terms of their dependence on r. Subsequently, Krivelevich [11] showed that in fact the dependence on r is (nearly) quadratic; that is, $\hat{R}(P_n,r) = r^{2+o_r(1)}n$. Furthermore, for $r \ge 6$ he also improved the lower bound and showed that $\hat{R}(P_n,r) \ge (r-2)^2 n + O(\sqrt{n})$. Here is the precise statement of his upper bound:

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Theorem 1 ([11])**.** *For any $C > 5$, $r \geq 2$, and all sufficiently large n we have*

$$\hat{R}(P_n, r) < 400^5 C r^{2+\frac{1}{C-4}} n.$$

It is straightforward to see that $C = C(r)$ that minimizes the upper bound in this theorem is of order $\log r$. As a result we get that $\hat{R}(P_n, r) = O((\log r) r^2 n)$. In this note, we give an alternative proof of this fact.

Theorem 2. *For any integer $r \geq 2$ and all sufficiently large n we have*

$$\hat{R}(P_n, r) < 600(\log r) r^2 n.$$

It will follow from the proof that the constant 600 is not optimal. Since we believe that the factor $\log r$ is not necessary, we do not attempt to optimize it.

2. Proof

Before we move to the proof of Theorem 2, we need one, straightforward, auxiliary result.

Proposition 3. *For any integer $r \geq 2$ there exists an integer $N = N(r)$ such that the following holds. For any integer $n \geq N$, there exists a graph $G = (V, E)$ such that*

(i) $|V| = 7rn$,
(ii) $500(\log r) r^2 n < |E| < 600(\log r) r^2 n$, *and*
(iii) *for every two disjoint sets $S, T \subseteq V$, $|S| = |T| = n$, the number of edges induced by $S \cup T$ with at least one endpoint in S is at most $70(\log r) n$.*

Proof. The proof is an easy application of random graphs. Recall that the *binomial random graph* $\mathcal{G}(n, p)$ is a distribution over the class of graphs with vertex set $[n]$ in which every pair $\{i, j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p, which may (and usually does) tend to zero as n tends to infinity. Furthermore, we say that events A_n in a probability space hold *asymptotically almost surely* (or *a.a.s.*), if the probability that A_n holds tends to 1 as n goes to infinity.

Fix any integer $r \geq 2$. It suffices to show that the random graph $G \in \mathcal{G}(7rn, p)$ with $p = 22(\log r)/n$ a.a.s. satisfies properties (ii) and (iii). (Property (i) trivially holds.) Indeed, if this is the case, then there exists an integer $N = N(r)$ such that the desired properties hold with probability at least $1/2$ for $G \in \mathcal{G}(7rn, p)$ for all $n \geq N$. This implies that for each $n \geq N$, there exists at least one graph with these properties.

Property (iii): Fix any two disjoint subsets $S, T \subseteq V$, both of cardinality n. Let $X_{S,T}$ be the random variable counting the number of edges induced by $S \cup T$ with at least one endpoint in S. Clearly, $X_{S,T}$ has the binomial distribution $\mathrm{Bin}\big(|S| \cdot |T| + \binom{|S|}{2}, p\big)$ with $\mathbb{E}(X_{S,T}) = (3/2 + o(1)) n^2 p = (33 + o(1))(\log r) n$. It follows from Chernoff's bound (see, for example, Corollary 21.7 in [10]) that

$$\Pr(X_{S,T} \geq 70(\log r)n) \leq \Pr(X_{S,T} \geq 2\mathbb{E}(X_{S,T})) \leq \exp(-\mathbb{E}(X_{S,T})/3) \leq \exp(-10.9(\log r)n).$$

Thus, the probability that there exist S and T such that $X_{S,T} \geq 70(\log r)n$ is, by the union bound, at most

$$\begin{aligned} \binom{7rn}{n}^2 \exp(-10.9(\log r)n) &\leq (7er)^{2n} \exp(-10.9(\log r)n) \\ &\leq \exp\Big(n\Big(2\log(7er) - 10.9 \log r\Big)\Big) = o(1), \end{aligned}$$

since $(7er)^2 < r^{10.9}$ for any $r \geq 2$. Property (iii) holds a.a.s.

Property (ii): This property is straightforward to prove. Note that $|E|$ is distributed as $\mathrm{Bin}\big(\binom{7rn}{2}, p\big)$ with $\mathbb{E}(|E|) = (539 + o(1))(\log r)r^2 n$. It follows immediately from Chernoff's bound that property (ii) holds a.a.s. The proof of the proposition is finished. □

Proof of Theorem 2. The proof is based on the depth first search algorithm (DFS), applied several times, and it is a variant of the previous approach taken in [7] where it was proved that $\hat{R}(P_n, r) \leq 33r4^r n$. Ben-Eliezer, Krivelevich and Sudakov [3] were the first to successfully apply the DFS algorithm to a Ramsey-type problem.

Fix $r \geq 2$ and suppose that n is sufficiently large so that Proposition 3 can be applied. Let $G = (V, E)$ be a graph satisfying properties (i)–(iii) from Proposition 3. We will show that $G \to (P_n)_r$ which implies the desired upper bound as $|E| < 600(\log r)r^2 n$ by property (ii). Consider any r-colouring of the edges of G. By an averaging argument, there is a colour (say blue) such that the number of blue edges is at least $|E(G)|/r$. For a contradiction, suppose that there is no monochromatic copy of P_n; in particular, there is no blue copy of P_n.

From now on, we restrict ourselves to the subgraph of G induced by blue edges, denoted $G_1 = (V_1 = V, E_1 \subseteq E)$. We perform the following algorithm on G_1 to construct a path P. Let v_1 be an arbitrary vertex of G_1, let $P = (v_1)$, $U = V \setminus \{v_1\}$, and $W = \emptyset$. If there exists an edge from v_1 to some vertex in U (say from v_1 to v_2), we extend the path as $P = (v_1, v_2)$ and remove v_2 from U. We continue extending the path P this way for as long as possible. Since there is no P_n in the blue graph, we must reach a point of the process in which P cannot be extended, that is, there is a path from v_1 to v_k ($k < n$) and there is no edge from v_k to U (including the case when U is empty). This time, v_k is moved to W and we try to continue extending the path from v_{k-1}, reaching another critical point in which another vertex will be moved to W, etc. If P is reduced to a single vertex v_1 and no edge to U is found, we move v_1 to W and simply re-start the process from another vertex from U, again arbitrarily chosen.

Observe that during this algorithm there is never an edge between U and W. Moreover, in each step of the process, the size of U decreases by 1 or the size of W increases by 1. The algorithm ends when U becomes empty and all vertices from P are moved to W. However, we will finish it prematurely, distinguishing $7r$ phases; phase i starts with graph $G_i = (V_i, E_i)$ and ends when for the first time $|W| = n$. Before we move to the next phase, we set $S_i = W$, $T_i = V(P)$, and F_i to be all edges incident to W. Then, we set $V_{i+1} = V_i \setminus W$ and $G_{i+1} = G_i[V_{i+1}]$, the graph induced by V_{i+1} (in other words, G_{i+1} is formed from G_i by removing vertices from W together with F_i, all edges incident with them). Phase i ends now and we move to phase $i + 1$ where we run the algorithm on G_{i+1}.

We make a few important observations. Note that, by property (i), $|V| = |V_1| = 7rn$ so the last phase, phase $7r$, finishes with $U = \emptyset$ and $T_{7r} = \emptyset$. As a result, family $(F_i : 1 \leq i \leq 7r)$ is a partition of E_1. By construction, $|S_i| = n$ for all i and, since there is no path on n vertices in G_1 (and so also in any G_i), $|T_i| < n$ for all i. Hence, $|F_i| < 70(\log r)n$ by property (iii). Putting these things together and using property (ii) in the very last inequality, we get the desired contradiction:

$$|E|/r \leq |E_1| = |F_1| + |F_2| + \cdots + |F_{7r}| \leq 7r \cdot 70(\log r)n < 500(\log r)rn < |E|/r.$$

The proof is finished. □

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