BURNING NUMBER OF GRAPH PRODUCTS

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Abstract. Graph burning is a deterministic discrete time graph process that can be interpreted as a model for the spread of influence in social networks. The burning number of a graph is the minimum number of steps in a graph burning process for that graph. In this paper, we consider the burning number of graph products. We find some general bounds on the burning number of the Cartesian product and the strong product of graphs. In particular, we determine the asymptotic value of the burning number of hypercube graphs and we present a conjecture for its exact value. We also find the asymptotic value of the burning number of the strong grids, and using that we obtain a lower bound on the burning number of the strong product of graphs in terms of their diameters. Finally, we consider the burning number of the lexicographic product of graphs and we find a characterization for that.

1. Introduction

Graph burning is a graph process that models the spread of influence in social networks and was introduced in [3, 4, 8]. Here is the definition of this process which is defined on the node set of a simple finite graph. There are discrete time-steps (or rounds) and initially all nodes are unburned. In the first round, we choose one node that catches fire. At the beginning of every round t ($t \geq 2$), the fire spreads from the set of burning nodes to their unburned neighbours. Then we choose one node and start the fire there, unless the node is already on fire. (Of course, choosing a node that is already on fire is usually suboptimal but we allow this to avoid complications with situations in which no unburned node is available.) Throughout the process, each node is either *burned* or *unburned*. Once a node is burned it remains in that state until the end of the process. The process ends at the end of round T when all nodes are burning.

Suppose that we burn a graph G in k steps in a burning process. For $1 \leq i \leq k$, the node x_i that we choose to burn directly in the i-th step of this process is called the i-th *fire source*. The sequence $(x_1, \ldots, x_k)$ is called a *burning sequence* for G. The *burning number* of a graph G, written by $b(G)$, is the length of a shortest burning sequence for G. Such a burning sequence is called an *optimum* burning sequence for G. For example, it is easy to see that $b(C_4) = 2$; the sequence (v_1, v_3) is an optimum burning sequence for C_4, as shown in Figure 1. The red nodes and edges demonstrates the fire spread from v_1 and the blue node is the fire started

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at v_3. The burning number can be used as a measure for the speed of spreading fire on the node set of graphs.

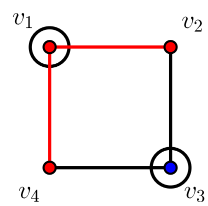

FIGURE 1. An optimum burning sequence for C_4.

Given two graphs G and H, one can create a new graph on the node set $V(G) \times V(H)$. There are several different ways to define the connections (or the edges) of such a graph, and they have been studied well in the theory of graphs; see [6]. Since the burning number is a relatively new parameter, it is natural to consider the burning number of graph products. Several facts and bounds on the burning number of graphs are given in [1, 4, 8]. It is shown in [2, 8] that the graph burning problem is **NP**-complete even for trees and path-forests. Some probabilistic results on the burning number of graphs, and some random variations of graph burning are presented in [7, 8]. In this paper, we consider the burning number of graph products and its relation to the burning number of the initial graphs.

2. Preliminaries

We first present some terminology, and then we review some known facts about graph burning and the burning number that are needed throughout the paper. For a node v in a graph G, the *eccentricity* of v is defined as $\max\{d(v,u) : u \in V(G)\}$. The *radius* of G, denoted by $\text{rad}(G)$, is the minimum eccentricity of a node in G. The *center* of G is the set of the nodes in G with minimum eccentricity. The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum eccentricity over the node set of G. For a positive integer k, the *k-th closed neighbourhood* of node v, denoted by $N_k[v]$, is defined to be the set $\{u \in V(G) : d(u,v) \le k\}$. We sometimes use the notation $N_k^G[v]$ to emphasize that we consider the k-th closed neighbourhood of node v in a specified graph G.

The *Cartesian product* of two graphs G and H, denoted by $G \square H$, is the graph with node set $V(G) \times V(H)$, in which two nodes (u_1, v_1) and (u_2, v_2) are adjacent if and only if, either $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $u_1 u_2 \in E(G)$ and $v_1 = v_2$. The *strong product* of two graphs G and H, denoted by $G \boxtimes H$, is the graph with node set $V(G) \times V(H)$, in which two nodes (u_1, v_1) and (u_2, v_2) are adjacent if and only if $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$. It is known that $d_{G \boxtimes H}((u_1, v_1), (u_2, v_2)) = \max\{d_G(u_1, u_2), d_H(v_1, v_2)\}$ (see, for example, [6]). By definition, we get immediately that $G \square H \subseteq G \boxtimes H$.

The *lexicographic product* of two graphs G and H, denoted by $G \circ H$, is the graph with node set $V(G) \times V(H)$, in which two nodes (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. In other words, $G \circ H$ is isomorphic to the graph that is constructed by replacing each node u_i in G by a copy of H, called H_i, and then adding all edges uv, where $u \in V(H_i)$, $v \in V(H_j)$, and u_iu_j is an edge in G. Namely, for $1 \leq i \leq |V(G)|$, $V(H_i) = \{(u_i, v) : v \in V(H)\}$. If $d_G(u_i, u_l)$ and $d_H(v_j, v_s)$ are finite (that is, u_i, u_j belong to the same connected component of G and v_j, v_s are in the same component of H), then for the nodes (u_i, v_j) and (u_l, v_s) in $G \circ H$, the following holds: if $u_i \neq u_l$, then

$$d_{G\circ H}\left((u_i, v_j), (u_l, v_s)\right) = d_G(u_i, u_l);$$

if $u_i = u_l$ and $v_j \neq v_s$, then

$$d_{G\circ H}\left((u_i, v_j), (u_l, v_s)\right) = \min\{2, d_H(v_j, v_s)\}.$$

For more on graph products see, for example, [6].

A subgraph H of a graph G is called an *isometric subgraph* if for every pair of nodes u and v in H, we have that $d_H(u, v) = d_G(u, v)$. For example, a subtree of a tree is an isometric subgraph. Also, if G is a connected graph and P is a shortest path connecting two nodes of G, then P is an isometric subgraph of G.

Here are some facts about the burning number from [3, 8] that we need for proving the results in this paper. From the definition of the burning process we can easily conclude the following lemma which is equivalent to Lemma 1 in [4].

Lemma 1. *A sequence $(x_1, x_2, \ldots, x_k)$ forms a burning sequence for a graph G if and only if*

$$N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \ldots \cup N_0[x_k] = V(G) \tag{1}$$

and

$$N_{k-2}[x_1] \cup N_{k-3}[x_2] \cup \ldots \cup N_0[x_{k-1}] \neq V(G).$$

The above lemma shows that a graph burning process for G is in fact a problem of covering the node set of G.

Theorem 2 ([8])**.** *If for some $t \leq k$, $C_1, C_2, \ldots, C_t$ is a collection of connected subgraphs in a graph G with radii $k-1, k-2, \ldots, k-t$, respectively, which cover all nodes of G, then $b(G) \leq k$.*

Indeed, (1) is satisfied with x_i being a center of subgraph C_i (that is, any vertex x_i such that all vertices of C_i are at distance at most $k-i$ from x_i.)

Note that if H is a spanning subgraph of a graph G, then $b(G) \leq b(H)$ (see [4, 8]).

Theorem 3 ([4])**.** *Suppose that H is an isometric subgraph of a graph G such that, for any node $x \in V(G) \setminus V(H)$, and any positive integer r, there exists a node $f_r(x) \in V(H)$ for which $N_r[x] \cap V(H) \subseteq N_r^H[f_r(x)]$. Then we have that $b(H) \leq b(G)$.*

Theorem 4 ([4])**.** *For any isometric subtree H of a graph G, we have that $b(H) \leq b(G)$.*

Theorem 5 ([4])**.** *A graph G of order n satisfies $b(G) = 2$ if and only if G is of order at least 2, and has maximum degree $n-1$ or $n-2$.*

Theorem 6 ([4])**.** *For a path P_n on n nodes, we have that $b(P_n) = \lceil n^{1/2} \rceil$.*

3. Main results

In this section, we present our main results as follows. We first state two simple general bounds on the burning number of the Cartesian product and the strong product of graphs. We then find the asymptotic value of the burning number of hypercube graphs, and we state a conjecture on the exact value. We also consider the asymptotic value of the burning number of strong grids; using that we obtain a lower bound on the burning number of the strong product of graphs in terms of their diameters. We finish this paper by characterizing the burning number of the lexicographic product of graphs.

We start with bounds on the burning number of the Cartesian product and the strong product of two graphs.

Theorem 7. *If G and H are two connected graphs, then we have that*

$$\max\{b(G), b(H)\} \leq b(G \boxtimes H) \leq b(G \square H) \leq \min\{b(G) + \mathrm{rad}(H), b(H) + \mathrm{rad}(G)\}.$$

Proof. First, note that $G \square H$ is a spanning subgraph of $G \boxtimes H$. Thus,

$$b(G \boxtimes H) \leq b(G \square H).$$

For proving the lower bound of $b(G \boxtimes H)$, note that each of G and H is an isometric subgraph of $G \boxtimes H$ that satisfies the conditions in Theorem 3. To see this, suppose that $V(G) = \{u_1, \ldots, u_m\}$ and $V(H) = \{v_1, \ldots, v_n\}$, for some positive integers m and n. Let u_k, where $1 \leq k \leq m$, be a central node in G. By definition of the strong product, the subgraph of $G \boxtimes H$ induced by the set of the nodes $\{(u_k, v_i) : 1 \leq i \leq n\}$ is isomorphic to H. We denote this subgraph by H_k. We claim that, for any node (u_i, v_j) in $G \boxtimes H$ and a positive integer r, we have that

$$N_r^{G \boxtimes H}[(u_i, v_j)] \cap H_k \subseteq N_r^{H_k}[(u_k, v_j)].$$

To see this, let $(u_k, v_l) \in V(H_k)$, where $1 \leq l \leq n$, with $d_{G \boxtimes H}\left((u_k, v_l), (u_i, v_j)\right) \leq r$. Then we have that

$$\begin{aligned} r \geq d_{G \boxtimes H}\left((u_k, v_l), (u_i, v_j)\right) &= \max\{d_G(u_k, u_i), d_H(v_l, v_j)\} \\ &\geq d_H(v_l, v_j) \\ &= d_{G \boxtimes H}\left((u_k, v_l), (u_k, v_j)\right). \end{aligned}$$

Hence, H is an isometric subgraph of $G \boxtimes H$ that satisfies the conditions in Theorem 3. Similarly, by symmetry of the strong product, we can conclude that

G is also an isometric subgraph of $G \boxtimes H$ that satisfies the conditions in Theorem 3. Therefore, we have that

$$b(G \boxtimes H) \geq \max\{b(G), b(H)\}.$$

Now, for proving the upper bound of $b(G\Box H)$, let $r = \mathrm{rad}(G)$, $s = b(H)$, and $(x_1, \ldots, x_s)$ be an optimum burning sequence for H. Suppose that u_k is a central node for G, and H_k is the subgraph of $G\Box H$ that is isomorphic to H corresponding to u_k, as defined above. For $1 \leq i \leq s$, let $y_i = (u_k, x_i)$. We can easily see that $\{N_{r+s-i}[y_i]\}_{i=1}^{s}$ forms a covering for the node set of $G\Box H$. Thus, by Theorem 2, we conclude that $b(G\Box H) \leq s + r$. By symmetry, we have that $b(G\Box H) \leq b(G) + \mathrm{rad}(H)$ and so the proof is finished. □

Note that the lower bound in Theorem 7 is achieved by $K_2\Box P_n$, where $n \in \{k^2+1, k^2+2 : k \in \mathbb{N}\}$. Also, it is achieved by $K_2 \boxtimes P_n$, where n is a square number. The upper bound is tight if G is any graph of radius one and H is a path of square order. For example, let $G = P_3$ and $H = P_4$. Then by Theorem 5, we can show that $b(G\Box H) > 2$. On the other hand, by Theorem 7, we have that $b(G\Box H) \leq 3$. Therefore, we conclude that $b(P_3\Box P_4) = 3$, which is suggested by the upper bound in Theorem 7.

The *hypercube graph*, or the *n-cube*, or the *n-dimensional hypercube*, denoted by Q_n, is the graph of order 2^n in which every node corresponds to a binary string of length n, and two nodes are adjacent if and only if their corresponding binary strings differ in exactly one bit. It is known (and easy to see) that $Q_0 = K_1$, $Q_1 = K_2$, and

$$Q_n = Q_{n-1}\Box K_2.$$

Moreover, the diameter of Q_n is n.

Suppose that we choose the nodes x_1 and x_2 in Q_n with $d(x_1, x_2) = n$, and we take $k = \lceil \frac{n}{2} \rceil + 1$. As Q_n is a node transitive graph, without loss of generality, we may assume that x_1 is the node that corresponds to the binary string with all zero bits, and x_2 corresponds to the binary string with all one bits. We then can easily see that $V(Q_n) = N_{k-1}[x_1] \cup N_{k-2}[x_2]$. Thus, by Theorem 2, we conclude that $b(Q_n) \leq k = \lceil \frac{n}{2} \rceil + 1$. We have the following conjecture for the optimum burning of the hypercube graphs.

Conjecture 8. *Let n be a positive integer, and $k = \lceil \frac{n}{2} \rceil + 1$. Then for the hypercube Q_n we have that $b(Q_n) = k$. Moreover, if n is even, then in any optimal burning sequence $(x_1, \ldots, x_k)$ for Q_n we must have $d(x_1, x_2) = n = \mathrm{diam}(Q_n)$.*

It is easy to check that the conjecture is true for $n \in \{1, 2, 3\}$ but it seems more challenging to prove it for any n. It is known that if $|n/2 - k| = o(n^{2/3})$, then

$$\binom{n}{k} \sim \frac{2^n}{\sqrt{\frac{1}{2}n\pi}} \mathrm{e}^{-\frac{(n-2k)^2}{2n}}; \qquad (2)$$

see, for example, Section 5.4 in [5]. We use this to find a slightly weaker lower bound on the burning number of the hypercube graph Q_n. This lower bound leads to an asymptotic result for the burning number of the hypercube graph Q_n.

Theorem 9. *For the hypercube graph Q_n, we have that*

$$b(Q_n) \sim n/2.$$

Proof. We will prove this by showing that

$$\frac{n}{2} + 1 - \sqrt{n \log n} < b(Q_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{3}$$

As we mentioned earlier, by burning two nodes $x_1, x_2 \in V(Q_n)$ with $d(x_1, x_2) = n$ in the first and second steps of a burning process for Q_n, we will have every node burning at time $t = \lceil \frac{n}{2} \rceil + 1$. Therefore, $b(Q_n) \leq \lceil \frac{n}{2} \rceil + 1$.

Now, assume that $k = \lceil \frac{n}{2} + 1 - c\sqrt{n \log n} \rceil$, where c is a constant that will be determined later on in the proof. We want to show that burning Q_n in k steps is asymptotically impossible. Clearly, the number of nodes in the r-th closed neighbourhood of any node in Q_n equals $\sum_{i=0}^{r} \binom{n}{i}$. Now, suppose that $(x_1, \ldots, x_k)$ is a sequence of nodes in Q_n. The total number of nodes that can be covered by $\bigcup_{i=1}^{k} N_{k-i}[x_i]$ is at most

$$\sum_{i=1}^{k} |N_{k-i}[x_i]| = \sum_{i=1}^{k} \sum_{j=0}^{k-i} \binom{n}{j} \leq \sum_{i=1}^{k} \sum_{j=0}^{k-i} \binom{n}{k-1} \leq k^2 \binom{n}{k-1},$$

since $\binom{n}{j}$ is an increasing function of j for $0 \leq j \leq k-1 \leq n/2$.

By (2), we have that

$$k^2 \binom{n}{k-1} \sim \frac{n^2}{4} \cdot \frac{2^n}{\sqrt{\frac{1}{2} n\pi}} \mathrm{e}^{-\frac{(n-2k+2)^2}{2n}} = O\left(n^{3/2} 2^n \mathrm{e}^{-2c^2 \log n}\right),$$

which is of order $o(2^n)$ if, for example, $c = 1 > \frac{\sqrt{3}}{2} \approx 0.866$. Thus, the lower bound in (3) holds and the proof is finished. □

We now consider the burning number of a $m \times n$ *strong grid*, that is, the strong product of P_m and P_n. Figure 2 shows an example of a 3×3 strong grid. Note that the r-th closed neighbourhood (in a strong grid G) of a node x (that is at distance at least r from the border of the grid) induces a smaller $(2r+1) \times (2r+1)$ strong grid centred at x. We call such an induced subgraph on precisely $(2r+1)^2$ nodes a *square of radius* r. We have the following theorem about the burning number of strong grids. The technique used in proving this theorem is similar to the technique used for finding the burning number of the Cartesian grids presented in [7].

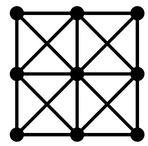

FIGURE 2. A 3×3 strong grid.

Theorem 10. *Let G be a $m \times n$ strong grid with $1 \leq m \leq n$, where $m = m(n)$ is a function of n. Then we have the following asymptotic results:*

$$b(G) = \begin{cases} \Theta(n^{1/2}) & \text{if } m = O(n^{1/2}) \\ (1+o(1))(\frac{3}{4})^{1/3}(mn)^{1/3} & \text{if } m = \omega(n^{1/2}). \end{cases}$$

Proof. First, we show a lower bound by applying Lemma 1 as follows. Suppose that $(x_1, \ldots, x_k)$, $k = b(G)$, is an optimal burning sequence for G. Thus, every node in G must be in the $(k-i)$-th neighbourhood of a node x_i, for some $1 \leq i \leq k$. Therefore, it follows from (1) that

$$\begin{aligned} mn = |V(G)| &\leq |N_{k-1}[x_1]| + |N_{k-2}[x_2]| + \cdots + |N_0[x_k]| \\ &\leq \sum_{i=1}^{k} (2i-1)^2 = \frac{k(2k-1)(2k+1)}{3} = \frac{4k^3}{3} - \frac{k}{3} < \frac{4k^3}{3}, \end{aligned}$$

and so

$$b(G) = k > (3/4)^{1/3}(mn)^{1/3}. \tag{4}$$

This bound is used when $m = \omega(n^{1/2})$. On the other hand, if $m = O(n^{1/2})$, then we use the fact that the path P_n is an isometric subtree of G. It follows from Theorem 4 that

$$b(G) \geq b(P_n) \geq n^{1/2}, \tag{5}$$

and hence the lower bounds are proved.

Now, let us move to the upper bounds. If $m < c_1\sqrt{n}$ for some constant c_1, then by Theorem 6 we may burn the path P_n on the top border of G in $\lceil \sqrt{n} \rceil$ steps. Since in this case every node in G is within distance $c_1\sqrt{n}$ from some node on this path, after at most $c_1\sqrt{n}$ additional steps all nodes in G must be burned. Therefore, we have that $b(G) = O(\sqrt{n})$ and the claimed upper bound in this case is proved.

It remains to concentrate on the case $m = \omega(n^{1/2})$. Let $\alpha = \alpha(n) = m/n^{1/2}$; note that $1 \ll \alpha \leq n^{1/2}$ as $m \leq n$. In order to avoid boundary effects, it will be convenient to consider an infinite grid that a finite grid G is part of. We are going to present a way to cover nodes of G with a family of squares of successive radii that are placed on some vertical strips. The radii of the squares are going to range from k_1 to some k_2 which will turn out to be at most k_3, where

$$
\begin{aligned}
k_1 &= \frac{(mn)^{1/3}}{\alpha^{1/6}} = n^{1/2}\alpha^{1/6} \quad \text{and} \\
k_3 &= \left(\frac{3}{4}\right)^{1/3} (mn)^{1/3} \left(1 + \frac{1}{\alpha^{1/6}}\right) = \left(\frac{3}{4}\right)^{1/3} n^{1/2}\alpha^{1/3} \left(1 + \frac{1}{\alpha^{1/6}}\right).
\end{aligned}
$$

For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible in the asymptomatic calculations we will make. Moreover, each radius in this range will appear exactly once in the proposed covering. By Theorem 2, we will get that $b(G) \leq k_2 + 1 \leq k_3 + 1 \sim \left(\frac{3}{4}\right)^{1/3} (mn)^{1/3}$, and the proof will be finished.

As already mentioned, in order to cover nodes of G with a family of squares, it is natural to arrange the squares by putting them in vertical strips without making an effort to avoid overlaps between the squares. The first strip, S_1, of width $2r_1+1 = 2k_1+1$ will be covered (from top to bottom) with disjoint squares of radii $r_1, r_1+1, \ldots, r_2-1$ for some $r_2 > r_1$. The squares will be put on the grid so that their right borders coincide with the border of the strip. On the other hand, as the radii are increasing, the squares will reach nodes away from the left border and so they will intersect with the squares that will be used to cover the next neighbouring strip. Such nodes will be called *overlapping*. Moreover, some part of the last square (of radius $r_2 - 1$) might fall outside of G (but, of course, be part of an infinite grid). Nodes that are covered but are not part of G will be called *wasted*. Once the first strip is covered, we move on to the next strip, S_2, of width $2r_2 + 1$, and proceed as before using squares of radii between r_2 and $r_3 - 1$, etc. An example of a layout of this covering is presented in Figure 3. We see a $m \times n$ strong grid G with black nodes and edges in the plane (with white background). The strong grid G is covered with blue squares of successive radii between r_1 (the radius of the first square on the top right corner of G) and r_ℓ (the radius of the last square on the left side of G). The centre node of each blue square is shown in red (representing a fire source). The grey rectangles represent the overlapping areas.

Our goal is to show that the total number of overlapping and wasted nodes is negligible comparing to the number of nodes of the finite grid G. Using the described strategy, we partition the strong grid G (from right to left) into some vertical strips $S_1, S_2, \ldots, S_\ell$ (for some positive integer ℓ), in which the radius of S_i is r_i. The increasing sequence $k_1 = r_1 < r_2 < \ldots < r_\ell < r_{\ell+1}$ is a function of n and m and is defined recursively. Note that the last square used in this covering has radius $k_2 = r_{\ell+1} - 1$, and so it follows from Theorem 2 that $b(G) \leq r_{\ell+1}$. It remains to show that $r_{\ell+1} - 1 \leq k_3$.

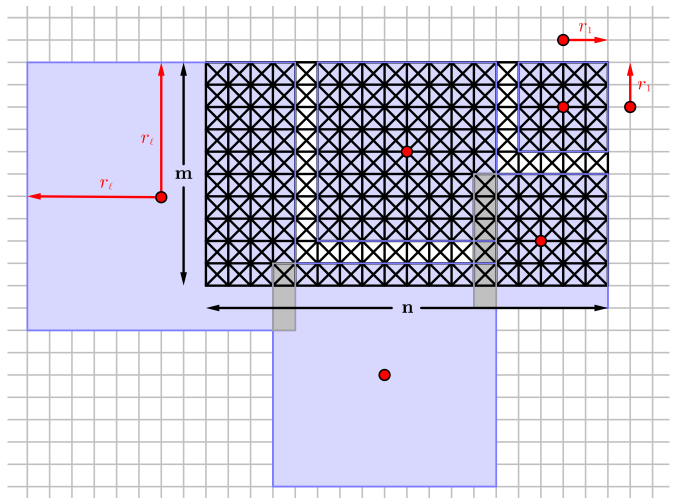

FIGURE 3. A covering of the $m \times n$ strong grid G.

It should be possible to estimate ℓ and r_i's but it seems rather complicated and quite tedious. Instead, for a contradiction, let us suppose that it is impossible to cover the grid with squares of radii between k_1 and k_3 (the way we described above); that is, we suppose that $k_2 > k_3$. In other words, we suppose that the family of squares of radii $k_1, \dots, k_3$ does not cover all nodes of G. Since G has mn nodes that are only partially covered by the family of squares of radii $k_1, \dots, k_3$, we must have

$$\sum_{i=k_1}^{k_3} (2i+1)^2 - E < mn,$$

where E is the number of nodes that are either wasted or overlapping. We are going to estimate E to get the desired contradiction.

Let us focus on any strip S_i ($1 \le i \le \ell - 1$) of width $2r_i + 1$ and length m (the last strip, S_ℓ, will be considered separately). Since the side length of each square is at least $2r_i + 1 \ge 2k_1 + 1$, the total number of squares used to cover S_i is at most

$$\left\lceil \frac{m}{2k_1+1} \right\rceil \le \frac{m}{2k_1} + 1$$

and hence we get $r_{i+1} \le r_i + m/(2k_1) + 1$. The number of overlapping nodes in the union of squares used to cover S_i that intersect with S_{i+1} is at most O_i, where

$$O_i = \Big(\big(2(r_{i+1} - 1) + 1\big) - \big(2r_i + 1\big) \Big) m \le \frac{m^2}{k_1}.$$

Since the radius of every square in the covering of S_i is at most $r_{i+1} - 1$, it is easy to see that the number of wasted nodes is at most W_i, where

$$W_i = \Big(2(r_{i+1} - 1) + 1\Big)^2 \le (2k_3 + 1)^2.$$

It follows that the total number of overlapping or wasted nodes in the covering of the first $\ell - 1$ strips is at most

$$E_1 = \sum_{i=1}^{\ell-1} (O_i + W_i) \le \left(\frac{m^2}{k_1} + (2k_3+1)^2 \right) \ell.$$

Moreover, since the width of every strip S_i, $1 \le i \le \ell - 1$, is at least $2k_1 + 1$, we have

$$\ell \le \frac{n}{2k_1+1} \le \frac{n}{2k_1},$$

and so we conclude that

$$E_1 \le \left(\frac{m^2}{k_1} + (2k_3+1)^2 \right) \frac{n}{2k_1}.$$

Now, let us focus on the last strip, S_ℓ. Since it is the last strip, there are no overlapping nodes. On the other hand, it might happen that almost all nodes associated with S_ℓ are wasted (we use E_2 to denote the number of them). We will use a trivial upper bound for E_2:

$$E_2 \le m(2r_{\ell+1}+1) \le m(2k_3+1).$$

Combining the obtained bounds we get that the total number of overlapping or wasted nodes can be estimated to be at most

$$\begin{aligned} E = E_1 + E_2 &\le \left(\frac{m^2}{k_1} + (2k_3+1)^2 \right) \ell + m(2r_{\ell+1}+1) \\ &\le \left(\frac{m^2}{k_1} + (2k_3+1)^2 \right) \frac{n}{2k_1} + m(2k_3+1) \\ &= \left(n^{1/2}\alpha^{11/6} + O(n\alpha^{2/3}) \right) \frac{n^{1/2}}{2\alpha^{1/6}} + O(mn^{1/2}\alpha^{1/3}) \\ &\le \left(n\alpha^{5/6} + O(n\alpha^{2/3}) \right) \frac{n^{1/2}}{2\alpha^{1/6}} + O(mn/\alpha^{2/3}) \sim \frac{mn}{2\alpha^{1/3}}. \end{aligned}$$

Hence, the number of nodes of the grid covered is at least

$$\begin{aligned} \sum_{i=k_1}^{k_3} (2i+1)^2 - E &= \frac{4(k_3+1)^3}{3} - \frac{k_3+1}{3} - \frac{4k_1^3}{3} + \frac{k_1}{3} - E \\ &> \frac{4k_3^3}{3} - \frac{4k_1^3}{3} - E \\ &= mn\left(1 + \frac{1}{\alpha^{1/6}}\right)^3 - O\left(\frac{mn}{\alpha^{1/2}}\right) - O\left(\frac{mn}{\alpha^{1/3}}\right) \\ &= mn + (1+o(1))\frac{3}{\alpha^{1/6}} mn, \end{aligned}$$

which is larger than mn, the order of G. This contradicts our assumption that $k_2 > k_3$, and the proof is finished. □

We now proceed with another lower bound on the burning number of the strong product of two connected graphs. By combining the following theorem and Theorem 10, we can find a lower bound on the burning number of the strong product of graphs in terms of their diameters.

Theorem 11. *Let G and H be two connected graphs with diameters d_1 and d_2, respectively. Suppose that P is a shortest path between two nodes of distance d_1 in G, and Q is a shortest path between two nodes of distance d_2 in H. Then we have that*

$$b(G \boxtimes H) \geq b(P \boxtimes Q).$$

Proof. To prove this, we will show that the subgraph $P \boxtimes Q$ of $G \boxtimes H$ satisfies the condition in Theorem 3. We first need to prove that $P \boxtimes Q$ is an isometric subgraph of $G \boxtimes H$. Let x, z be two nodes in P and y, w be two nodes in Q. Note that $d_P(x, z) = d_G(x, z)$ and $d_Q(y, w) = d_H(y, w)$. Thus, we have that

$$\begin{aligned} d_{P\boxtimes Q}\left((x, y), (z, w)\right) &= \max\{d_P(x, z), d_Q(y, w)\} \\ &= \max\{d_G(x, z), d_H(y, w)\} = d_{G\boxtimes H}\left((x, y), (z, w)\right). \end{aligned}$$

Now, we will show that the second condition in Theorem 3 holds as follows. Let (u, v) be a node in $V(G \boxtimes H) \setminus V(P \boxtimes Q)$, and r be a positive integer. Also, assume that A is the smallest connected subgraph of P that contains $N^r_G[u] \cap P$. Clearly, A must be a subpath of P with end points in $N^r_G[u]$. Similarly, we define B to be the smallest subpath of Q that contains $N^r_H[v] \cap Q$.

Note that both A and B are of radius at most r (as path graphs). To see this, assume that x and y are the end points of A (as a path). Then we have that

$$d_A(x, y) = d_P(x, y) = d_G(x, y) \leq d_G(x, u) + d_G(u, y) \leq r + r = 2r.$$

Thus, A is a path of length at most $2r$, and consequently, the radius of A must be at most r. Similarly, we can prove that B is a path of radius at most r.

Now, we can easily see that $A \boxtimes B$ is a subgrid of $P \boxtimes Q$ with radius at most r, and its centre consists of either a single node or two adjacent nodes, or four nodes that are mutually adjacent. Namely, if the centres of A and B are C_1 and C_2, respectively, then the centre of $A \boxtimes B$ is $C_1 \boxtimes C_2$. Moreover, note that $N^r_{G\boxtimes H}[(u, v)] \cap (P \boxtimes Q)$ is a subgraph of $A \boxtimes B$. To show this, assume that (x, y) is a node in $N^r_{G\boxtimes H}[(u, v)] \cap (P \boxtimes Q)$. This implies that $x \in P$, $y \in Q$, and $d_{G\boxtimes H}\left((u, v), (x, y)\right) \leq r$. Thus, $\max\{d_G(u, v), d_H(v, y)\} \leq r$. This means that $d_G(u, v) \leq r$ and $d_H(v, y) \leq r$. Hence, we must have that $x \in A$ and $y \in B$. Therefore, $(x, y) \in A \boxtimes B$.

Thus, the above arguments imply that

$$N^r_{G\boxtimes H}[(u, v)] \cap (P \boxtimes Q) \subseteq (A \boxtimes B) = N^r_{A\boxtimes B}[(u_0, v_0)],$$

where (u_0, v_0) is a central node in $A \boxtimes B$. Hence, the conditions in Theorem 3 hold. Therefore, we conclude that

$$b(G \boxtimes H) \geq b(P \boxtimes Q),$$

and the proof is finished. □

The following corollary follows immediately from Theorem 11 and the proof of Theorem 10 (see (4) and (5)).

Corollary 12. *Let G and H be two connected graphs with diameters d_1 and d_2, respectively, such that $d_1 \leq d_2$. Then we have that*

$$b(G \boxtimes H) > \max\{d_2^{1/2}, (3/4)^{1/3}(d_1 d_2)^{1/3}\}.$$

We now move on to the burning number of $G \circ H$. Note that when G is a single node, $G \circ H$ is isomorphic to H and clearly, $b(G \circ H) = b(H)$. Hence, we consider $b(G \circ H)$ for G being of order at least two. Here we give a simple characterization of $b(G \circ H)$ for G being connected and of order at least two.

Theorem 13. *Let G be a connected graph of order at least two and H be any graph. Then we have that*

$$b(G) \leq b(G \circ H) \leq b(G) + 1.$$

Moreover, $b(G \circ H) = b(G)$ if and only if one of the following conditions holds.

(i) $b(H) = 1$ or equivalently $H = K_1$.

(ii) $b(H) = 2$ and G has an optimum burning sequence $(x_1, \ldots, x_k)$ such that one of the neighbours of x_k is burned in step $k - 1$.

(iii) G has an optimum burning sequence $(x_1, \ldots, x_k)$ such that each of x_{k-1} and x_k has a neighbour that is burned in step $k - 1$.

Proof. Suppose that $V(G) = \{u_1, \ldots, u_n\}$, and $V(H) = \{v_1, \ldots, v_m\}$, where m and n are two positive integers. Note that the subgraph of $G \circ H$ induced by the nodes $\{(u_1, v_1), (u_2, v_1), \ldots, (u_n, v_1)\}$ is isomorphic to G. We call this subgraph G_1, and we claim that G_1 is an isometric subgraph of $G \circ H$ that satisfies the conditions in Theorem 3: to show this, suppose that (u_i, v_1) and (u_j, v_1) are two nodes in G_1, where $1 \leq i < j \leq n$. As we mentioned in Section 2,

$$d_{G \circ H}\left((u_i, v_1), (u_j, v_1)\right) = d_G(u_i, u_j) = d_{G_1}(u_i, u_j).$$

Hence, G_1 is an isometric subgraph of $G \circ H$. Now, suppose that (u_i, v_j) is a node in $G \circ H$, with $j \neq 1$, and r is a positive integer. We can easily see that $N_r^{G \circ H}[(u_i, v_j)] \cap V(G_1) \subseteq N_r^{G_1}[(u_i, v_1)]$. Thus, the claim is true. Therefore, by Theorem 3, we conclude that

$$b(G \circ H) \geq b(G_1) = b(G),$$

thus proving the first inequality.

Now, for proving the second inequality, assume that $b(G) = k$. Let $(x_1, \ldots, x_k)$ be an optimum burning sequence for G. For $1 \leq i \leq n$, assume that H_i denotes the subgraph of $G \circ H$ that is isomorphic to H and corresponds to the node $u_i \in G$. That is, $V(H_i) = \{(u_i, v) : v \in V(H)\}$. Now, for $1 \leq j \leq k$, take $y_j = (x_j, v_1)$. Clearly, $(y_1, \ldots, y_k)$ forms a burning sequence for G_1. Thus, by the end of the k-th step, each H_i contains the burning node (u_i, v_1) (at least). Note that $\{V(H_i)\}_{i=1}^n$ forms a partition for the node set of $G \circ H$. Since by assumption, G_1 is connected and contains at least two nodes, by definition of the lexicographic product, we can see that after burning the sequence $(y_1, \dots, y_k)$ every H_i has a neighbour (that is, a node that is adjacent to all nodes in H_i) that is burning. Therefore, $\{N_{k+1-j}[y_j]\}_{j=1}^k$ forms a covering for the node set of $G \circ H$. Hence, by Theorem 2, we conclude that $b(G \circ H) \leq k + 1 = b(G) + 1$.

We now claim that $b(G \circ H) = b(G) = k$ if and only if one of the conditions (i), (ii), and (iii) in theorem's statement holds. We first assume that one of the three conditions holds:

Case **(i)**: If $H = K_1$, then $G \circ H = G$, and the statement of the theorem is clearly true.

Case **(ii)**: Assume now that $b(H) = 2$, and there is an optimum burning $(x_1, \dots, x_k)$ for G such that one of the neighbours of x_k is burned in step $k - 1$. Since $b(H) = 2$, then H is of order at least two and we have two possibilities: either H is of radius one, or there are two non-adjacent nodes $x, y \in V(H)$ such that $V(H) = N[x] \cup \{y\}$. If the former holds, then without loss of generality, assume that v_1 is a central node in H. If the latter holds, then without loss of generality assume that v_1 and v_2 are two non-adjacent nodes in H such that $V(H) = N[v_1] \cup \{v_2\}$. In the first case let $(y_1, \dots, y_k)$ be the sequence in which $y_j = (x_j, v_1)$, for $1 \leq j \leq k$. In the second case, let $(y_1, \dots, y_k)$ be the sequence in which $y_j = (x_j, v_1)$, for $1 \leq j \leq k - 1$, and $y_k = (x_{k-1}, v_2)$. We claim that in both cases, $(y_1, \dots, y_k)$ is a burning sequence for $G \circ H$. Now, from the assumptions, we can see that in both cases, every H_i will have a neighbour that is burning after burning the nodes $(y_1, \dots, y_{k-1})$. Hence, every node in $G \circ H$ will be within distance one from a burning node by the end of step $k - 1$. Therefore, $(y_1, \dots, y_k)$ is a burning sequence for $G \circ H$.

Case **(iii)**: This time we assume that G has an optimum burning sequence $(x_1, \dots, x_k)$ such that each of x_{k-1} and x_k has a neighbour that is burned in step $k - 1$. If $b(H) \leq 2$, then by the above cases, we conclude that $b(G \circ H) = b(G)$. If $b(H) \geq 3$, then let $(y_1, \dots, y_k)$ be the sequence in which $y_j = (x_j, v_1)$, for $1 \leq j \leq k$, and with v_1 being any node in H. Then similar to Case (ii) it is easy to see that $(y_1, \dots, y_k)$ is a burning sequence for $G \circ H$.

We next aim at proving the converse direction. That is, we suppose $b(G \circ H) = b(G)$, and our goal is to show that one of the three conditions must hold. On a high level, our goal is to provide an algorithm that modifies, if necessary, the burning sequence of $G \circ H$ in such a way that all fire sources (except for possibly one) belong to G_1 and each of the H_i (except for possibly one) contains at most one fire source. After having performed these changes, we are able to show that one of the three conditions given in the statement of the theorem must be fulfilled.

We explain the idea now in more detail. First note that since G is of order at least two, by definition of the lexicographic product, for $2 \le j \le n$, every node in H_j is adjacent to every node in H_1. Thus, the distance between every pair of distinct nodes $x, y \in V(H_i)$, where $1 \le i \le n$, is at most two in $G \circ H$. Moreover, by definition of the lexicographic product and the facts we mentioned in Section 2, it is easy to see that $d(x,z) = d(y,z)$ for every node $z \in (G \circ H) \setminus H_i$. Thus, for every $r \ge 2$ and $s \ge 0$ with $s < r$,

$$N_s[x] \subseteq N_r[x] = N_r[y]. \tag{6}$$

Hence, by Lemma 1, we conclude that for $1 \le i \le n$, there can be at most three fire sources in H_i. More precisely, there are five possibilities: either $y_j, y_{j+1}, y_{j+2} \in V(H_i)$, or only $y_j, y_{j+1} \in V(H_i)$, or only $y_j, y_{j+2} \in V(H_i)$, or only $y_j \in V(H_i)$, or there is no fire source in H_i. As mentioned before, we will now modify the burning sequence (if necessary) to find a new burning sequence for $G \circ H$ in which all fire sources, except possibly y_k, are in G_1, and there is at most one fire source from each H_i, for $1 \le i \le n$, except possibly the one that contains y_{k-1}. This is made precise by the following algorithm (we denote the copy of H in $G \circ H$ that contains y_i by H_{y_i}, for $1 \le i \le k$):

Algorithm 14. *Suppose that* $(y_1, \dots, y_k)$ *is a burning sequence for* $G \circ H$. *We then perform the following steps.*

Stage 1. *For* $1 \le j \le k-1$, *perform the following steps:*

Stage 1.1. *If* $y_j \notin G_1$, *that is,* $y_j = (u_i, v_\ell)$ *for some* $1 \le i \le n$ *and* $2 \le l \le m$, *then do the following: if* $y_{j+1} = (u_i, v_1)$, *let* $y_j = (u_i, v_1)$ *and* $y_{j+1} = (u_i, v_\ell)$ *(switching* y_j *and* y_{j+1}*). If* $y_{j+2} = (u_i, v_1)$, *then let* $y_j = (u_i, v_1)$ *and* $y_{j+2} = (u_i, v_\ell)$ *(switching* y_j *and* y_{j+2}*). If* $(u_i, v_1) \notin \{y_{j+1}, y_{j+2}\}$, *then let* $y_j = (u_i, v_1)$.

Stage 1.2. *If* $y_j, y_{j+1} \in V(H_i)$ *for some* $1 \le i \le n$ *and* $1 \le j \le k-2$, *then select* $x \in V(G_1) \setminus \big((\cup_{i=1}^{j} N_{j-i}[y_i]) \cup (\cup_{i=1}^{j} H_{y_i}) \big)$ *with* $x \ne y_\ell$, *for* $\ell \ge j+2$.
Set $y_{j+1} = x$ *in the sequence* $(y_1, \dots, y_k)$.

If $y_{k-1}, y_k \in V(H_i)$ *for some* $1 \le i \le n$ *and* $y_k \in \big(\cup_{i=1}^{k-1} N_{k-i}[y_i]\big)$, *then select* $x \in V(G_1) \setminus \big((\cup_{i=1}^{k-1} N_{k-1-i}[y_i]) \cup (\cup_{i=1}^{k-1} H_{y_i}) \big)$.
Set $y_k = x$ *in the sequence* $(y_1, \dots, y_k)$.

Stage 1.3. *If* $y_j, y_{j+2} \in V(H_i)$ *for some* $1 \le i \le n$, *then select* $x \in V(G_1) \setminus \big((\cup_{i=1}^{j+1} N_{j+1-i}[y_i]) \cup (\cup_{i=1}^{j+1} H_{y_i}) \big)$ *with* $x \ne y_\ell$, *for* $\ell \ge j+3$.
Set $y_{j+2} = x$ *in the sequence* $(y_1, \dots, y_k)$.

Stage 2. *If* $y_{k-1} \in H_i$ *and* $y_k = (u_j, v_\ell)$ *for some* $1 \le i, j \le n$, *with* $i \ne j$, *and* $2 \le \ell \le m$, *then set* $y_k = (u_j, v_1)$ *in the sequence* $(y_1, \dots, y_k)$.
Return the sequence $(y_1, \dots, y_k)$.

We first show the following:

Claim. *Algorithm 14 returns a burning sequence for $G \circ H$ in which the first $k-1$ fire sources are all in G_1, and there is at most one fire source in each H_i, $1 \leq i \leq n$, except possibly one of them.*

Proof of the claim. In Stage 1, we go through the first $k-1$ fire sources, and in every step we make sure that y_j is in G_1. Each time that we find a second fire source chosen from the same H_i, where $1 \leq i \leq n$, we replace that fire source with a new one that is not in H_i, using the assumptions. As we show in the sequel, this happens for all y_j's except possibly y_k. In Stage 2, we replace y_k by a node from G_1 if possible. To prove this, we consider all possibilities as follows.

Part 1. For $1 \leq j \leq k-2$, suppose that $y_j = (u_i, v_\ell) \in H_i$, for some $1 \leq i \leq n$ and $2 \leq \ell \leq m$ (we consider $j = k-1$ in Part 2). Note that in this part $k-j \geq 2$. If y_{j+1} (or y_{j+2}, respectively) is in H_i, then by (6), we conclude that $N_{k-j-1}[y_{j+1}] \subseteq N_{k-j}[y_j] = N_{k-j}[(u_i, v_1)]$ (or $N_{k-j-2}[y_{j+2}] \subseteq N_{k-j}[y_j] = N_{k-j}[(u_i, v_1)]$, respectively). Therefore, by switching the nodes suggested in Stage 1.1, and by Lemma 1, the new sequence $(y_1, \ldots, y_k)$ is still a burning sequence for $G \circ H$.

Also, in Stage 1.2 we can find a node $x \in V(G_1) \setminus \big(\big(\cup_{i=1}^{j} N_{j-i}[y_i]\big) \cup \big(\cup_{i=1}^{j} H_{y_i}\big)\big)$ (similarly in Stage 1.3, $x \in V(G_1) \setminus \big(\big(\cup_{i=1}^{j+1} N_{j+1-i}[y_i]\big) \cup \big(\cup_{i=1}^{j+1} H_{y_i}\big)\big)$, respectively) with $x \neq y_\ell$, for $\ell \geq j+2$ ($\ell \geq j+3$, respectively), since otherwise, the sequence $(y_1, \ldots, y_j)$ (or $(y_1, \ldots, y_{j+1})$, respectively) must be a burning sequence for G_1 of length less than k, which is a contradiction. Thus replacing y_{j+1} (y_{j+2}, respectively) by x satisfies the conditions in Lemma 1. Hence, by the changes suggested to the sequence $(y_1, \ldots, y_k)$ in Stage 1.2 (Stage 1.3, respectively) of Algorithm 14, for $1 \leq j \leq k-2$, y_j is the only fire source in H_{y_j}, and it is in G_1.

Part 2. For $j = k-1$, if $y_{k-1}, y_k \in H_i$, for some $1 \leq i \leq n$, then note that all nodes in $V(H_i)$ must be burned by the end of k-th step in the burning process: that is, only one step after burning y_{k-1}. Thus, either $N[y_{k-1}] \cup \{y_k\} = V(H_i)$ (equivalently, $b(H) = 2$), or there must be a neighbour of y_k in $(G \circ H) \setminus H_i$ (which is by definition of $G \circ H$ adjacent to all nodes in $V(H_i)$) that is burned in step $k-1$.

Case 2.1. If $N[y_{k-1}] \cup \{y_k\} = V(H_i)$ (that is, $b(H) = 2$) and no neighbour of y_k in $(G \circ H) \setminus H_i$ is burned in step $k-1$, then either $y_k \in N[y_{k-1}]$ (in case H is of radius one), or $d(y_k, y_{k-1}) = 2$. In both cases we must have already $y_{k-1} = (u_i, v_1)$, as otherwise, $V(H_i)$ cannot be burned by the end of k-th step. Thus, nothing happens in Stage 1.1 of these two cases. We now consider the two mentioned possibilities as follows.

Case 2.1.1. If $y_k \in N[y_{k-1}]$ (H is of radius one), then there must be a node $x \in V(G_1) \setminus \big(\big(\cup_{i=1}^{k-1} N_{k-1-i}[y_i]\big) \cup \big(\cup_{i=1}^{k-1} H_{y_i}\big)\big)$, since otherwise, the subsequence $(y_1, \ldots, y_{k-1})$ must be a burning sequence for G_1, which is a contradiction. Hence, by Lemma 1, in Stage 1.2, we are allowed to replace y_k by x. Thus, in this case $y_k \in G_1$. Therefore, all fire sources in the sequence $(y_1, \ldots, y_k)$ (that is returned as the output of Algorithm 14) are in G_1 and each H_j, $1 \leq j \leq n$, contains at most one fire source.

Case 2.1.2. If $d(y_k, y_{k-1}) = 2$, then $y_k \notin \left(\cup_{i=1}^{k-1} N_{k-i}[y_i]\right)$. Therefore, to maintain the conditions in Lemma 1, we are not allowed to replace y_k by any other node in Stage 1.2 and Stage 2. Thus, in such a case Algorithm 14 returns a burning sequence in which all the first $k-1$ fire sources are in G_1, and each H_j, $1 \leq j \leq n$, except H_i that contains y_{k-1} and y_k, contains at most one fire source.

Case 2.2. If there is a node in $(G \circ H) \setminus H_i$ that is adjacent to all nodes in $V(H_i)$ and is burned in step $k-1$, then it means $y_{k-1}, y_k \in \left(\cup_{j=1}^{k-2} N_{k-j}[y_j]\right)$. Thus, by Lemma 1, y_{k-1} and y_k can be any two nodes in $V(H_i)$, and therefore, the switching suggested in Stage 1.1 gives a new burning sequence for $G \circ H$. Also, in this case, there must be a node $x \in V(G_1) \setminus \left(\left(\cup_{j=1}^{k-1} N_{k-1-j}[y_j]\right) \cup \left(\cup_{j=1}^{k-1} H_{y_j}\right)\right)$, since otherwise, the subsequence $(y_1, \ldots, y_{k-1})$ must be a burning sequence for G_1, which is a contradiction. Hence, by Lemma 1, in Stage 1.2 we are allowed to replace y_k by $x \in G_1$. Thus in this case, Algorithm 14 returns a burning sequence $(y_1, \ldots, y_k)$ in which all fire sources are in G_1 and each H_j, $1 \leq j \leq n$, contains at most one fire source.

Finally, in Stage 2, we check if we can choose $y_k \in G_1$ in case y_k and y_{k-1} are not in the same H_i. Therefore, after performing Algorithm 14 we have a burning sequence $(y_1, \ldots, y_k)$ for $G \circ H$ in which all fire sources, except possibly y_k, are in G_1, and there is at most one fire source in H_i, for $1 \leq i \leq n$, except possibly the one that contains y_{k-1} and y_k. This finishes the proof of the claim. □

We now come back to the proof of the converse direction. If H is of order one, the result is trivial, so we may assume that $b(H) \geq 2$. There are now two possibilities: either $b(H) = 2$ or $b(H) \geq 3$.

Case 1. If $b(H) = 2$, then either H is of radius one, or $V(H) = N[v_1] \cup \{v_2\}$ for non-adjacent nodes v_1 and v_2.

Case 1.1. If H is of radius one, then assume that v_1 is a central node in H. By Case 2.1.1 of the proof of the claim, we know that all fire sources in the sequence $(y_1, \ldots, y_k)$ are in G_1 (and in distinct H_j's). Thus, by Lemma 1, we have that

$$V(G_1) \subseteq V(G \circ H) = \cup_{j=1}^{k} N_{k-j}[y_j] = \left(\cup_{j=1}^{k-1} N_{k-j}[y_j]\right) \cup \{y_k\}.$$

It implies that $(y_1, \ldots, y_k)$ is also a burning sequence for G_1. Let $y_k = (u_i, v_1) \in H_i$, for some $1 \leq i \leq n$. We know that $V(H_i) \setminus \{y_k\}$ contains a node $y = (u_i, v_\ell)$ (since $b(H) = 2$). Note that y_k is the only fire source in H_i. Thus, y must have been burned only in step k: indeed, otherwise, y would have received the fire in an earlier step from a burning neighbour x in $(G \circ H) \setminus H_i$. But then, by definition of the lexicographic product, x must be also a neighbour of y_k, which is a contradiction (since y_k also must have received the fire from that node before step k). Now, in order to have y burned in step k, there must be a neighbour of y in $(G \circ H) \setminus H_i$ that is burned in step $k-1$. Let x be such a neighbour, and assume that $x = (u_j, v_r) \in H_j$, where $1 \leq j \leq n$ with $j \neq i$, and $1 \leq r \leq m$. If x is a fire source, then obviously $x \in G_1$. If x is not a fire source, then for some $1 \leq s \leq k-2$, $x \in N_{k-s}[y_s]$, since it is burned in step $k-1$. By definition of the lexicographic product, this implies that every node in H_j including (u_j, v_1) must be also burned in step $k-1$. Note that $(u_j, v_1) \in G_1$ is also a neighbour of y_k (by definition of the lexicographic product). Hence, $(y_1, \ldots, y_k)$ is a burning sequence for G_1 in which y_k has a neighbour (in G_1) that is burned in step $k-1$, and condition (ii) is satisfied in this case.

Case 1.2. If $V(H) = N[v_1] \cup \{v_2\}$ for non-adjacent nodes v_1 and v_2, then assume that $y_{k-1} \in H_i$, where $1 \leq i \leq n$. There are two possibilities: either y_k is in H_i, or not.

Case 1.2.1. If $y_k \in H_i$, then note that by Case 2.1.2 of the proof of the claim, the first $k-1$ fire sources are in G_1 and each H_j, $1 \leq j \leq n$, except H_i, contains at most one fire source. By Lemma 1,

$$V(G_1) \subseteq V(G \circ H) = \cup_{j=1}^{k} N_{k-j}[y_j] = \left(\cup_{j=1}^{k-1} N_{k-j}[y_j]\right) \cup \{y_k\}.$$

Now, note that there must be a node $x \in V(G_1) \setminus \left(\left(\cup_{j=1}^{k-1} N_{k-1-j}[y_j]\right) \cup \left(\cup_{j=1}^{k-1} H_{y_j}\right)\right)$, as otherwise, the sequence $(y_1, \ldots, y_{k-1})$ is a burning sequence for G_1, which is a contradiction. Let $x \in H_\ell$, where $1 \leq \ell \leq n$, and $\ell \neq i$. Since x must be burned in k-th step, there must be a neighbour of x in $\cup_{j=1}^{k-1} N_{k-1-j}[y_j]$ (the set of nodes that are burning by the end of step $k-1$) that is burned in step $k-1$. Thus,

$$V(G_1) \subseteq \left(\cup_{j=1}^{k-1} N_{k-j}[y_j]\right) \cup \{x\}.$$

Hence, by Lemma 1, we conclude that the sequence $(y_1, \ldots, y_{k-1}, x)$ forms a burning sequence for G_1. Moreover, a neighbour of x is burned in step $k-1$, and condition (ii) is satisfied in this case as well.

Case 1.2.2. If $y_k \notin H_i$, then observe that in this case $(y_1, \ldots, y_k)$ is a burning sequence for $G \circ H$ such that all y_j's are in G_1. Thus, with an analogous argument to the case where H is of radius one we conclude that $(y_1, \ldots, y_k)$ is also a burning sequence for G_1 in which y_k has a neighbour (in G_1) that is burned in step $k-1$, and condition (ii) is also satisfied in this case.

Case 2. If $b(H) \geq 3$, then let $y_{k-1} \in H_i$, for some $1 \leq i \leq n$. We have then two possibilities: either $y_k \in H_i$, or not. However, by Case 2.2 of the proof of the claim, we can see that in such a case having $y_k \in H_i$ in the sequence returned by Algorithm 14 is impossible. That is, all fire sources in the sequence $(y_1, \ldots, y_k)$ are in G_1, and there is at most one fire source in each H_j, $1 \leq j \leq n$. Moreover, there must be a neighbour of y_{k-1} in $(G \circ H) \setminus H_i$ that is burned in step $k-1$. Let x be such a neighbour, and assume that $x = (u_j, v_r) \in H_j$, where $1 \leq j \leq n$ with $j \neq i$, and $1 \leq r \leq m$. If x is a fire source, then obviously $x \in G_1$. If x is not a fire source, then for some $1 \leq s \leq k-2$, $x \in N_{k-s}[y_s]$, since it is burned in step $k-1$. By definition of the lexicographic product, this implies that node

(u_j, v_1) in $H_j \cap G_1$ must be also burned in step $k-1$. Note that (u_j, v_1) is also a neighbour of y_{k-1}. Hence y_{k-1} has a neighbour in G_1 that is burned in step $k-1$. Further, by Lemma 1, $V(G_1) \subseteq V(G \circ H) = \cup_{j=1}^{k} N_{k-j}[y_j]$. Thus, the sequence $(y_1, \ldots, y_k)$ is a burning sequence for G_1 as well. Now, assume that $y_k \in H_\ell$, where $1 \leq \ell \leq n$ (with $i \neq \ell$). Again note that H_ℓ is of order at least two, and therefore, $V(H_\ell) \setminus \{y_k\}$ contains a node y. Using an analogous argument to the one in the case where H is of radius one, we conclude that y_k must have a neighbour in G_1 that is burned in step $k-1$. Thus $(y_1, \ldots, y_k)$ is a burning sequence for G_1 in which each of y_{k-1} and y_k has a neighbour that is burned in step $k-1$. Hence, condition (iii) is satisfied in this case, and the proof is finished. □

We finish with an example showing how to apply Thorem 13. By Theorem 6, we know that $b(P_n) = \lceil \sqrt{n} \rceil = k$. By the proof of Theorem 6 given in [8], we can easily see that P_n has an optimum burning sequence $(x_1, \ldots, x_k)$ such that one of the neighbours of x_k is burned before the k-th step if and only if $n < k^2$. Moreover, if $n \leq k^2 - 2$, then P_n has an optimum burning sequence $(x_1, \ldots, x_k)$ such that each of x_k and x_{k-1} has a neighbour that is burned in step $k-1$. Thus, by Theorem 13, $b(P_n \circ H) = b(P_n) = \lceil \sqrt{n} \rceil$ if and only if one of the following conditions holds:

(i) $H = K_1$.

(ii) $b(H) = 2$ and $n = k^2 - 1$.

(ii) $n \leq k^2 - 2$ and H is any graph.

If $n = k^2$, then in every optimum burning sequence $(x_1, \ldots, x_k)$ of P_n all neighbours of x_{k-1} and x_k are burned in the k-th step. Therefore, by Theorem 13, in such a case $b(P_n \circ H) = b(P_n) + 1 = k + 1$, with H being any graph of order at least two.

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