# Endogenous differentiation of consumer preferences under quality uncertainty in a SPA network

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**Abstract.** We study a duopoly market on which there is uncertainty of a product quality. Consumers adaptively learn about quality of products when they buy them (direct learning) or from other consumers with whom they are interacting in a social network modelled as a SPA graph (indirect learning). We show that quality uncertainty present in such a market leads to endogenous segmentation of consumers' preferences towards suppliers. Additionally, we show that in this setting, even if both companies have the same expected quality, the company with lower variance of quality will gain higher market share.

## 1 Introduction

In economics textbooks a discussion of network effects is usually limited to positive or negative externalities caused by preferential attachment process. In this paper we argue that below the surface of these evident and well known phenomena there is a layer of more subtle and less understood ones. Specifically the subject of our study is perception of product quality by consumers embedded in a network of social interactions induced by a SPA graph. We show how presence of quality uncertainty in this setting leads to diversification of quality expectations and endogenous differentiation of consumer preferences towards suppliers.

The significance of quality expectations for market mechanism has been originally revealed in the seminal paper of George Akerlof (1970). Using a market for second hand cars as an example, Akerlof has shown how presence of quality uncertainty leads to adverse selection of low quality products and degeneration or even failure of a market. More specifically, he considered a market comprised of two types of agents — car owners and potential car buyers. The first group was offering cars for sale, asking for prices reflecting their actual quality, but bids from the second group were based on the *average* quality of the cars on the market, as the true quality distribution was hidden from them. The mismatch of supply and demand discouraged owners of better than average cars causing them to withdraw from the market. This, in turn, triggered a feedback loop of gradual deterioration of quality of cars on sale until the market collapsed in the end.

The discovery of this apparently simple mechanism of adverse selection has proven to be one of the most fruitful insights in modern economics, spawning development of vast literature and helping to explain market phenomena as diverse as drastic loss of value suffered by brand-new vehicles on their first days of use, difficulties of elderly people or young motorcyclists to get insurance cover, dearth of credit markets in underdeveloped countries or high unemployment among minorities. Its profound influence was eventually recognized by awarding Akerlof and his collaborators the Nobel Memorial Prize in Economic Sciences in 2001. With such prominence and after nearly half a century of research, one would hardly expect any novelty on the topic. So the paper of Izquierdo and Izquierdo (2007) came as a surprise by providing a new insight into the phenomena and suggesting that influence of uncertain quality on markets is still not entirely understood.

Akerlof (1970) and his followers considered *information asymmetry* between the transacting parties to be the necessary prerequisite for a market degradation to occur. This assumption has been considered so fundamental that the 2001 Nobel Prize was awarded to Akerlof, Spence and Stiglitz "for their analyses of markets with asymmetric information." Quite unexpectedly therefore

Izquierdo and Izquierdo (2007) claimed that the same effect could be induced by quality uncertainty alone. To prove it they proposed an agent-based model in which consumers estimated quality of a product based upon experiences of their own and of their acquaintances. As they have demonstrated under the assumed adaptive quality estimation process the symmetry of supply and demand breaks down and market degenerates, down to a point of non-existence of equilibrium, in the same way as in the model of Akerlof (1970). The effect is more pronounced when there is only individual quality estimation, whereas social interaction mitigates it. In contrast to the model of Akerlof (1970) no a priori assumption of information asymmetry is required as it is replaced by endogenous differentiation of quality expectations in the population of consumers.

It may surprise at first that the mechanism described by Izquierdo and Izquierdo (2007) has not been revealed earlier during nearly half a century of research. However under scrutiny one will notice a fundamental difference in a way quality expectations are formed in the two models. Car buyers in Akerlof (1970) follow what is known as the rational expectations hypothesis (REH). In short REH assumes that agents know the "true" structural form of the data generating process parameters of which they estimate and their subjective expectations are consistent with this knowledge. This approach assumes car buyers to have precise knowledge of the average quality of cars on sale at any moment (although not on quality of each individual item). In contrast consumers of Izquierdo and Izquierdo (2007) follow the adaptive expectations hypothesis (AEH), which does not assume this accurate a priori knowledge. Instead, agents apply a simplistic first-order prediction error correction formula with exponentially decaying weights.

AEH and REH are the two extremes on the spectrum of expectation formulation methods. AEH as the earlier approach was commonly used in economics (for a classic example see Nerlove 1958) until the critique by Muth (1960) who shown its non-optimal statistical properties as the "backward-looking" biased estimator. Use of AEH was discouraged afterwards and replaced by REH as subsequently proposed by Muth (1961) and advocated by Lucas (1972) and Sargent (1973). The fact that REH has been integrated into the paradigm of the mainstream neoclassical economics does not discredit the adaptive approach nonetheless. In fact REH is often criticized for making too strong, psychologically unrealistic assumptions of agents rationality and their ability to perceive and process information (Evans and Honkapohja 2001). Another problem is that in many models REH results in multiple equilibria but does not indicate how to resolve the conundrum. For these reasons AEH is being actively researched as a viable alternative and many hybrid learning tactics combining AEH and REH are proposed (Frydman and Phelps 2013).

We hence find effects spotted by Izquierdo and Izquierdo (2007) as intriguing enough to deserve further exploration, although in a slightly modified setting. Both the discussed models required disgruntled agents, car sellers in Akerlof (1970) and consumers in Izquierdo and Izquierdo (2007), to retreat from the market for its contraction to occur. While this assumption could be justified in some circumstances, in many others it would be unrealistic. A person would rather look for substitutes than give up consumption altogether. Therefore in our model we assume a market with alternative suppliers of a homogeneous good and we allow consumers to switch suppliers to maximize satisfaction.

Note that although technically we consider a case of oligopoly, we are not interested in strategic interplay of suppliers. We use term "oligopoly" not in a classical sense but merely to signal that a key assumption of our model is ability of consumers to distinguish between multiple suppliers. The subject of our study is dynamics of consumer preferences under quality uncertainty and adaptive expectations. The ability of consumers to discriminate suppliers is a necessary prerequisite for it, but does not restrict the context to the textbook definition of oligopoly. Our conclusions are general and extend onto any market where consumers are able to perceive quality variability and discern suppliers, regardless of market power exercised by the latter. So they readily apply to monopolistic or perfect competition, as long as the key assumption of supplier distinction holds. In other words our aim is to study dynamics of quality expectations in isolation and we assume both supply and demand to be totally inelastic.

The remaining part of the paper is organized as follows. In Section 2 we describe the model of the market with quality uncertainty in detail. For better understating it will be presented in two variants for (a) finite and (b) continuous state space. In Section 3 we present results of the base analysis of an isolated consumer which do not take into account effects of social interaction. In Section 4 we extend the analysis with network effects induced by a socially realistic SPA connection graph. Section 5 presents the summary of the results and concludes.

## 2 Model

In this section, we define the model that we investigate. In order to validate robustness of presented results we present two variants of it. The first variant is a minimal specification that exhibits base properties we want to explore and better understand; it uses a finite state-space representation. The other variant has uncountable state-space and so it is more realistic, but obviously more challenging to handle.

### 2.1 Finite state-space model

Let $\mathcal{G} = (A, C)$ be a directed graph of connections between agents from set $A$. A connection $c \in C \subseteq A \times A$ is an ordered tuple $(a_1, a_2) \in C$ indicating that agent $a_1$ has influence on opinion of agent $a_2$. As mentioned, the graph is directed. It will be assumed at some point that $\mathcal{G}$ is a random geometric graph generated by the Spatial Preferential Attachment model that received some attention recently, see Section 4.1.

Suppose that there are two companies in the model, each providing some product or service. Customer buying product from company $i \in \{1, 2\}$ observes its value equal to a sample being random variable $Q_i$. It is assumed that samplings are independent. We take that $Q_i$ is defined as an identity on the probability space $(\Omega, 2^{\Omega}, P_i)$, where $\Omega = \{-1, 0, 1\}$. We will say that $-1$ is a bad value of the product, 0 is a normal value of the product and 1 is an excellent value of the product. For this probability space we naturally define a probability mass function $p_i \colon \Omega \to ]0, 1[$ (we assume that each state has strictly positive probability). In the model, we consider discrete time-steps. Each agent $a \in A$, in each time-step $t$, has evaluation of quality of company $i$ as $e^t_{a,i} \in \{-1, 0, 1\}$, i.e. the agent believes that the company has bad, normal or excellent quality product. A vector of evaluations of agent $a$ in time $t$ is denoted as $\mathbf{e}^t_a$.

We start with time-step $t = 0$. Assume that $e^0_{a,i}$ is a random i.i.d. drawn from $Q_i$. The dynamics of the model is defined by the following procedure:

1. increase time-step $t \leftarrow t + 1$;
2. select $a \in A$ uniformly at random;
3. agent $a$ selects company $i$ to buy from as a draw from random variable $S(\mathbf{e}^t_a)$ specified below;
4. agent $a$ observes quality of selected company $i$ as $q_i$ being a random sample from $Q_i$;
5. *individual learning*: agent $a$ updates her beliefs $e^t_{a,i}$ as a draw from random variable $\Gamma(e^t_{a,i}, q_i)$ where $\Gamma$ is specified below;
6. *social learning*: each agent $b$ for which $(a, b) \in C$ updates her beliefs $e^t_{b,i}$ as a draw from random variable $\Delta(e^t_{a,i}, q_i)$, where $\Delta$ is specified below.

Definition of random variable $S$ is the following:

– if $e^t_{a,i} = e^t_{a,j}$ then
$$\Pr(S = i) = \Pr(S = j) = 1/2;$$

– if $e^t_{a,i} > e^t_{a,j}$ then

$$\Pr(S = i) = \beta \text{ and } \Pr(S = j) = 1 - \beta, \text{ where } \beta \in [0.5, 1);$$

Definition of random variable $\mathit{\Gamma}$ is the following:

$$\Pr(\mathit{\Gamma} = q_i) = \gamma \text{ and } \Pr(\mathit{\Gamma} = e^t_{a,i}) = 1 - \gamma, \text{ where } \gamma \in (0, 1).$$

Definition of random variable $\mathit{\Delta}$ is the following:

$$\Pr(\mathit{\Delta} = q_i) = \delta \text{ and } \Pr(\mathit{\Gamma} = e^t_{a,i}) = 1 - \delta, \text{ where } \delta \in [0, 1).$$

### 2.2 Continuous state space

For simplicity, here we will discuss only the differences in specification of this model in comparison to the previous, finite state-space model. The model is analysed on the same graph $\mathcal{G}$ and has the same specification of dynamics. We provide new parameter names in this model, but there is a direct correspondence between these parameters and the parameters in finite state-space variant.

As before, we consider two companies. Customer buying a product from company $i$ observes its value equal to sample from random variable $Q_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$. The parameter pair $(\mu_i, \sigma_i)$ corresponds to pair $(p_i(-1), p_i(1))$ in the finite state space model.

We may assume that initially (that is, at time $t = 0$) agents have beliefs that are based on their first time purchases, i.e. $e^0_{i,a}$ is a sample from $Q_i$. Alternatively, we may assume that each agent start with a given value of $e^0_{i,a}$, say, $e^0_{i,a} = \mu_i$. The selection function $S(\mathbf{e}^t_a)$ is specified as:

$$S(\mathbf{e}^t_a) = \arg\max_i \{e^t_{i,a} + \varepsilon_i\},$$

where $\varepsilon_i \sim N(0, \omega^2)$ and are independent. Parameter $\omega$ corresponds to parameter $\beta$ in finite state-space model.

After company $i$ is selected agent $a$ observes a sample of quality $Q_i$ denoted as $q_i$.

Beliefs in the continuous state space model are deterministically updated. Individual learning is governed by the rule:

$$e^{t+1}_{i,a} \leftarrow (1 - \lambda_1)e^t_{i,a} + \lambda_1 q_i,$$

where $\lambda_1 \in (0, 0.5)$ is an individual learning parameter. Social learning follows the rule:

$$e^{t+1}_{i,b} \leftarrow (1 - \lambda_2)e^t_{i,b} + \lambda_2 q_i,$$

where $\lambda_2 \in [0, 0.5)$ is a social learning parameter. Observe that $\lambda_1$ corresponds to $\gamma$, and $\lambda_2$ to $\delta$ in the finite state-space model.

## 3 Baseline analysis without network effects

In this section, we investigate both models in a simple scenario where network does not affect the behaviour of agents (that is, $\delta = 0$ or $\lambda_2 = 0$, depending on the variant considered).

### 3.1 Finite state-space model

If $\delta = 0$, then our model can be thought of as a single customer model where an evolution of customer's state is given by a Markov process. The transition matrix $M_{9\times 9}$ can be explicitly analytically derived.[3] Observe that the process is irreducible and positive recurrent, given the

[3] We omit it in the text as it is large but easy to derive.

assumed domains of parameters. Therefore, it has a unique steady state $\pi$ that is a solution of the following system of equations:

$$\begin{cases} \pi^T(M - I) = \mathbf{0} \\ \pi^T\mathbf{1} = 1 \end{cases}$$

Additionally, we can observe that then $\gamma$ only influences the speed of convergence of the process as $M - I = \gamma X$, where $X$ does not depend on $\gamma$. Under such observations $\pi$ can be expressed as a function of five parameters: $\beta$, $p_1(-1)$, $p_1(1)$, $p_2(-1)$, and $p_2(1)$.

In order to assess the model we use the following metrics of the steady state:

- Mean evaluation of company $i$ by the customer: $E(e_{a,i})$;
- Probability that company $i$ is evaluated better than company $j$: $\Pr(e_{a,i} > e_{a,j})$;
- Probability that company $i$ is evaluated equally to company $j$: $\Pr(e_{a,i} = e_{a,j})$.

Since the model is symmetric with respect to companies 1 and 2, in the following analysis we concentrate on company 1. In Figures 1, 2 and 3 we show these metrics for the case when we assume that offers of companies are symmetric, i.e. $p_i(-1) = p_i(1)$, which means that $E(Q_i) = 0$ for both companies (the plots show averages over $\beta$ uniformly distributed in intervals specified in subplot captions). We argue that this case is interesting because it represents the situation where both companies are equally good but only differ in the dispersion of their qualities. In this text we will solely concentrate on this scenario.

One can conclude from plots that in the steady state:

F1) $\Pr(e_{a,i} = -1) > \Pr(e_{a,i} = 1)$ so $E(e_{a,i}) < 0$; on the average customers have negatively biased opinion; this bias is potentially significant and reaches $\approx -0.6$, when the range of possible results is $[-1, 1]$;

F2) $p_i(1) < p_j(1) \Rightarrow E(e_{a,i}) > E(e_{a,j})$; company with higher variance has lower market share; this is the crucial finding of no-network model: it pays off to give customers service with predictable quality;

F3) for high $\beta$, $p_1(1)$, $p_2(1)$ we have bimodality, i.e. $\min_{i\in\{1,2\}} \Pr(e_{a,i} > e_{a,3-i}) > \Pr(e_{a,1} = e_{a,2})$; most likely client has a clear preference for one product or the other.

As a side note (not investigated in detail in this paper), let us observe that the system exhibits significantly nonlinear behaviour; shape of relationships of measured quantities changes with $\beta$.

The key question raised in this paper is how network structure affects the results F1–F3 (i.e., what happens when $\delta = 0$ is replaced by $\delta > 0$). In particular:

- how does the in-degree of a given agent influence bias of her preferences;
- do customers that are connected in a graph have correlated preferences;
- how does $\delta$ influence bias in evaluation of performance of companies;
- how does $\delta$ influence the presence of bimodality of preferences.

### 3.2 Continuous state-space model

We move to continuous state-space model but continue investigating the variant with no network effects, that is, when $\lambda_2 = 0$. We present two approaches to highlight tools that can be used in such situations. The first one is asymptotic in nature and provides a statement that holds asymptotically almost surely (and so can be considered to be more rigorous). The second one is based on simulations and can be applied for finite (but usually large) number of agents (and so can be considered to be more realistic).

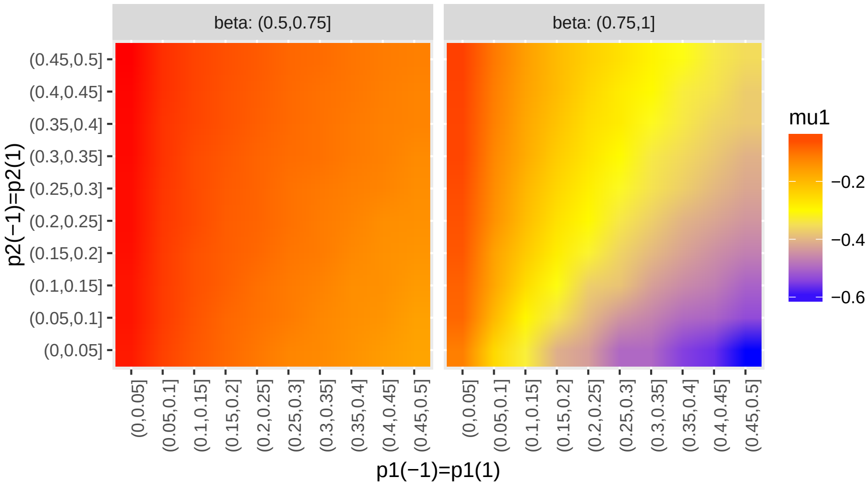


**Fig. 1.** $E(e_{a,1})$: mean of evaluation of company 1 for symmetric offers (mu1).

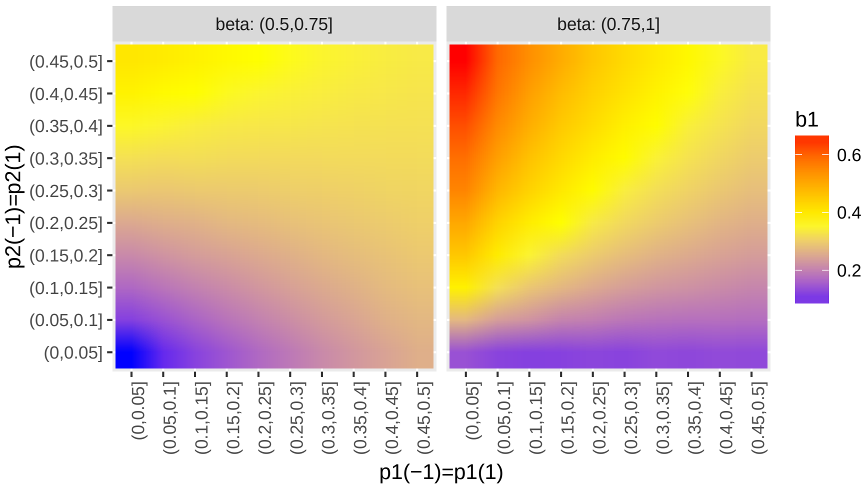


**Fig. 2.** $\Pr(e_{a,1} > e_{a,2})$: probability that company 1 has better evaluation than company 2 (b1).

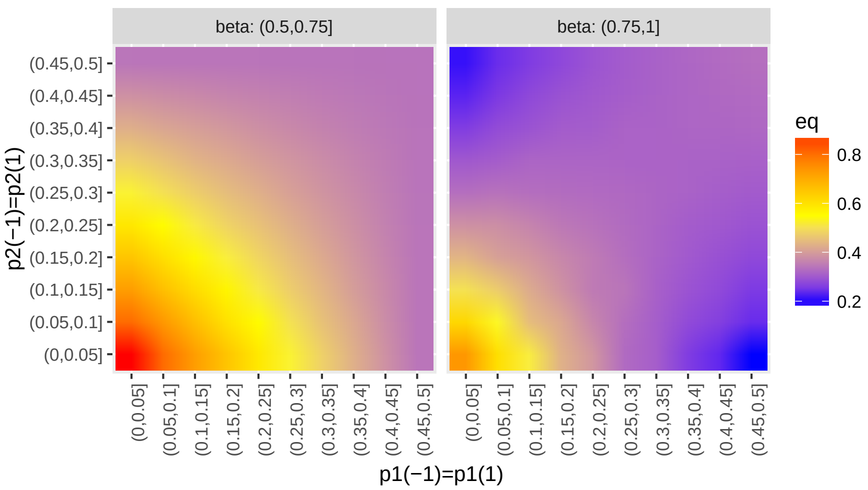


**Fig. 3.** $\Pr(e_{a,1} = e_{a,2})$: probability that companies are equally evaluated (eq).

**Differential equations approach** As typical in random graph theory, all results in this subsection are asymptotic; that is, for $n = |A|$ tending to infinity. We say that an event holds asymptotically almost surely (a.a.s.) if it holds with probability tending to one as $n \to \infty$.

The general setting that is used in the differential equation method (Wormald 1999) is a sequence of random processes indexed by $n$ (which in our case is the number of agents). The aim is to find asymptotic properties of the random process and the conclusion we aim for is that variables defined are well concentrated, which informally means that a.a.s. they are very close to certain deterministic functions. These functions arise as the solution to a system of ordinary first-order differential equations. One of the important features of this approach is that the computation of the approximate behavior of processes is clearly separated from the proof that the approximation is correct.

First, let us discretize the space of potential states agents can be in. Fix a real number $z > 0$, an integer $k$, and let us restrict ourselves to $(2k+1)$ values of possible believes for a company $c \in \{1,2\}$: $\mu_c - zk/k, \mu_c - z(k-1)/k, \ldots, \mu_c, \ldots, \mu_c + z(k-1)/k, \mu_c + zk/k$. Each time some belief is updated, it is immediately rounded up or down to the nearest possible value (for a given company $c$).

For $-k \le i, j \le k$, let $X_{i,j}(t)$ be a random variable counting the number of agents of type $(i,j)$ with belief about product 1 equal to $\mu_1 + zi/k$ and with belief about product 2 equal to $\mu_2 + zj/k$. Let $q(i,j)$ be the probability that agent of type $(i,j)$ buys product 1; that is,

$$\begin{aligned} q(i,j) &= P\Big(\mu_1 + zi/k + N(0,\omega) \ge \mu_2 + zj/k + N(0,\omega)\Big) \\ &= P\Big(N(0,2\omega) \ge (\mu_2 - \mu_1) + z(j-i)/k\Big). \end{aligned}$$

The probability that she buys product 2 is, of course, $q(j,i) = 1 - q(i,j)$. Now, let $r(s,i,\mu,\sigma^2)$ be the probability that an agent changes her believes from $\mu + zs/k$ to $\mu + zi/k$ after buying product with the corresponding quality distribution $N(\mu,\sigma^2)$ (and after rounding, of course); that is, for $-k < i < k$

$$\begin{aligned} r(s,i,\mu,\sigma^2) &= P\Big(\mu + \frac{z(i-1/2)}{k} \le (1-\lambda_1)\Big(\mu + \frac{zs}{k}\Big) + \lambda_1 N(\mu,\sigma^2) \le \mu + \frac{z(i+1/2)}{k}\Big) \\ &= P\Big(\frac{z(i - s(1-\lambda_1) - 1/2)}{k\lambda_1} \le N(\mu,\sigma^2) - \mu \le \frac{z(i - s(1-\lambda_1) + 1/2)}{k\lambda_1}\Big) \\ &= P\Big(\frac{z(i - s(1-\lambda_1) - 1/2)}{k\lambda_1} \le N(0,\sigma^2) \le \frac{z(i - s(1-\lambda_1) + 1/2)}{k\lambda_1}\Big). \end{aligned}$$

For the two extreme values ($i = -k$ and $i = k$) we have

$$\begin{aligned} r(s,-k,\mu,\sigma^2) &= P\Big(N(0,\sigma^2) \le \frac{z(i - s(1-\lambda_1) + 1/2)}{k\lambda_1}\Big) \\ r(s,k,\mu,\sigma^2) &= P\Big(\frac{z(i - s(1-\lambda_1) - 1/2)}{k\lambda_1} \le N(0,\sigma^2)\Big). \end{aligned}$$

Our goal is to estimate the conditional expectation $\mathbb{E}\left[X_{i,j}(t+1) - X_{i,j}(t) \mid \mathbf{X}\right]$ (given the set $\mathbf{X}$ of all variables $X_{i,j}(t)$). Note that an agent of type $(i,j)$ is selected with probability $X_{i,j}(t)/n$. Conditioning on this event, the probability she stays within this group is equal to

$$q(i,j) \cdot r(i,i,\mu_1,\sigma_1^2) + q(j,i) \cdot r(j,j,\mu_2,\sigma_2^2).$$

For $s \ne i$, an agent of type $(s,j)$ is selected with probability $X_{s,j}(t)/n$, and conditioning on that, she becomes of type $(i,j)$ with probability

$$q(s,j) \cdot r(s,i,\mu_1,\sigma_1^2).$$

Similarly, for $y \neq j$, an agent of type $(i,y)$ is selected with probability $X_{i,y}(t)/n$, and conditioning on that, she becomes of type $(i,j)$ with probability

$$q(y,i) \cdot r(y,j,\mu_2,\sigma_2^2).$$

It follows that

$$\begin{aligned}\mathbb{E}\left[X_{i,j}(t+1) - X_{i,j}(t) \mid \mathbf{X}\right] = &-\frac{X_{i,j}(t)}{n} \\ &+ \sum_{s=-k}^{k} \frac{X_{s,j}(t)}{n} q(s,j) r(s,i,\mu_1,\sigma_1^2) \\ &+ \sum_{y=-k}^{k} \frac{X_{i,y}(t)}{n} q(y,i) r(y,j,\mu_2,\sigma_2^2).\end{aligned}$$

For simplicity, as this is just an illustration of the general method, we may assume that, say, $X_{0,0} = n$, and other values are 0 (that is, initially every agent believes that product 1 has quality $\mu_1$ and product 2 has quality $\mu_2$). Any other scenario can be investigated the same way affecting only the initial value for the system of differential equations we are about to set up.

Now, one can scale everything down (both the time $n$ and the number of members of each group, $n$) to get the system of differential equations. Here, function $f_{i,j}(x)$ is used to model random variable $X_{i,j}(xn)/n$. We get the system of $(2k+1)^2$ equations: for $-k \leq i,j \leq k$,

$$\begin{aligned}f'_{i,j}(x) = &-f_{i,j}(x) \\ &+ \sum_{s=-k}^{k} f_{s,j}(x) q(s,j) r(s,i,\mu_1,\sigma_1^2) \\ &+ \sum_{y=-k}^{k} f_{i,y}(x) q(y,i) r(y,j,\mu_2,\sigma_2^2),\end{aligned}$$

with the initial value $f_{0,0}(0) = 1$ and $f_{i,j}(0) = 0$ if $|i| + |j| > 0$.

Finally, the differential equations method (introduced and developed by Wormald 1999) can be used to show that our random variables are well-concentrated around their expectations. Using the general purpose theorem (Theorem 5.1 in Wormald 1999), we get that a.a.s. for any $-k \leq i,j \leq k$, and any $t$, we have

$$X_{i,j}(t) = (1+o(1)) f_{i,j}(t/n) n.$$

In order for the discretized model to approximate with good accuracy the original, continuous model, one should take: i) $z$ large enough so that, for any company $c \in \{1,2\}$, in the original model, the number of agents that get belief below $\mu_c - z$ or above $\mu_c + z$ is negligible; ii) $k$ large enough to capture a large spectrum of beliefs. As a result, plotting all $(2k+1)^2$ functions is an impossible task but the following three functions should describe well the behaviour of the system:

$$\begin{aligned}f_{=}(x) &:= \sum_{i=-k}^{k} f_{i,i}(x) \quad \text{(fraction of agents that equally like both products)} \\ f_{>}(x) &:= \sum_{i=-k}^{k} \sum_{j=-k}^{i-1} f_{i,j}(x) \quad \text{(fraction of agents that like product 1 more)} \\ f_{<}(x) &:= \sum_{i=-k}^{k} \sum_{j=i+1}^{k} f_{i,j}(x) \quad \text{(fraction of agents that like product 2 more)}\end{aligned}$$

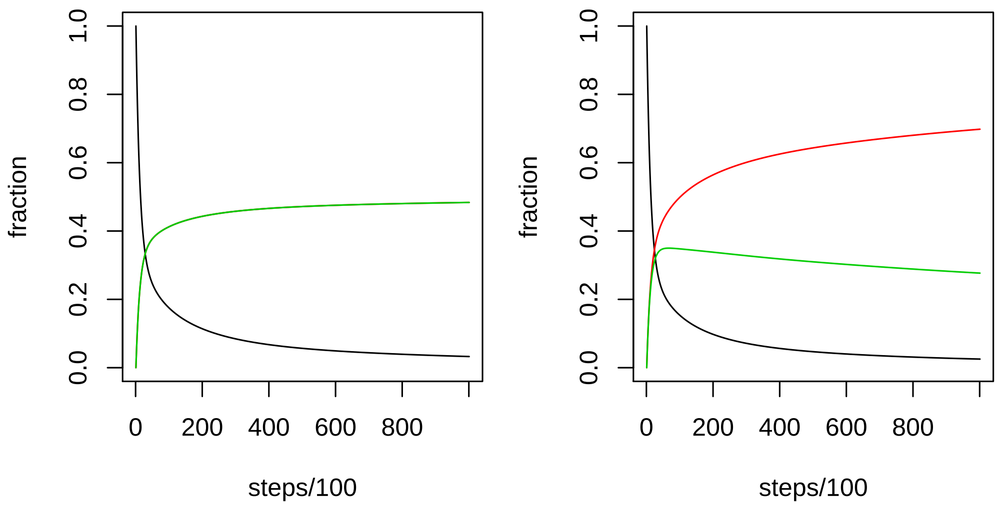


**Fig. 4.** Example of the dynamics of the model. Black line represents $f_=$, red line $f_>$, and green line $f_<$. On the left plot we consider a symmetric case: $\sigma_1 = \sigma_2 = 1$ (red and green lines overlap). On the right plot an asymmetric case is considered: $\sigma_1 = 1$ and $\sigma_2 = 1.25$. In both plots $\mu_1 = \mu_2$, $\lambda_1 = 0.5$ and $\omega = 0.1$. In approximation we used $z = 5$ and $k = 100$. Initial beliefs are equal to $\mu_c$, for both companies.

Figure 4 presents an example of the dynamics of the process. If $\sigma_2 > \sigma_1$, then the company with a product having lower variability gains higher market share. Interestingly, the transient behavior of this simulation is that initially company 2 gains market share, but then starts losing it as $f_=$ drops and $f_>$ continues to increase. As $k$ increases $f_=$ will tend to 0 in general. Economically this might have twofold implications: (1) it might be profitable to launch even a product that is known to be inferior than competition because profits that can be reaped in the initial period might justify it and (2) investors should look at initial success of a product with care as it might be only transient characteristic of a system.

Finally, let us stress again that we present an application of the differential equations method in a very simple setting but it can be easily generalized to more sophisticated scenarios. For example, when agents are not selected uniformly at random from set $A$ (see step 2.) but, instead, with probability that is affected by type $(i, j)$ a given agent is of. Or perhaps agents select company to buy from (see step 3.) with probability that depends on how many other agents have similar believes. In these examples, this method seems to be the only tool one can use. On the other hand, in our example, one can avoid using it as it is straightforward to predict (a.a.a., as always) how many agents at time $xn$ (for some constant $x$) made $\ell$ purchases ($\ell = 0, 1, \ldots$). Then, each agent can be investigated independently (based on the assigned value of $\ell$) using Markov processes, as for the finite state-space model (but with $(2k + 1)$ states instead of 9). Putting things together, we can calculate the expected number of agents of a given type and the concentration will follow from standard tools (such as Chernoff bound), since the corresponding events are independent.

**Simulation approach** The simulation was run for 1'000'000 iterations which was enough for it to reach steady-state. In Figures 5 and 6 we can observe that expected evaluation of company 1 is negative and that it decreases with $\sigma_1$ and increases with $\sigma_2$. Additionally increase of $\omega$ reduces those differences (as customers behave more randomly).

In Figure 7 we show distribution of beliefs of agents for $\sigma_1 = \sigma_2 = 1$, $\lambda_1 = 0.1$ and $\lambda_2 = 0$. For this parameterization we have that $E(e_{a,i}) \approx -0.27$ and correlation between $e_{a,1}$ and $e_{a,2}$ equals to approximately $-0.65$. The crucial thing is that as depicted on the plot we observe bimodality in the beliefs of agents — both in one belief and for combination of two beliefs.

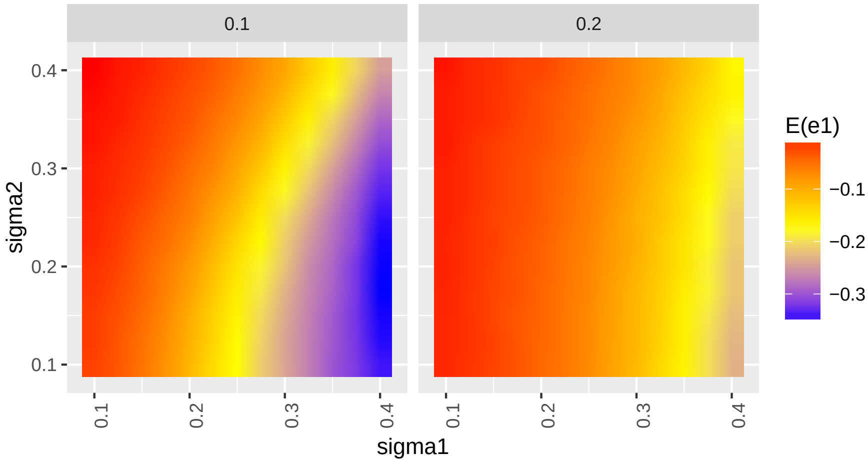


**Fig. 5.** $E(e_{a,1})$: mean evaluation of company 1 (for two values of $\omega$).

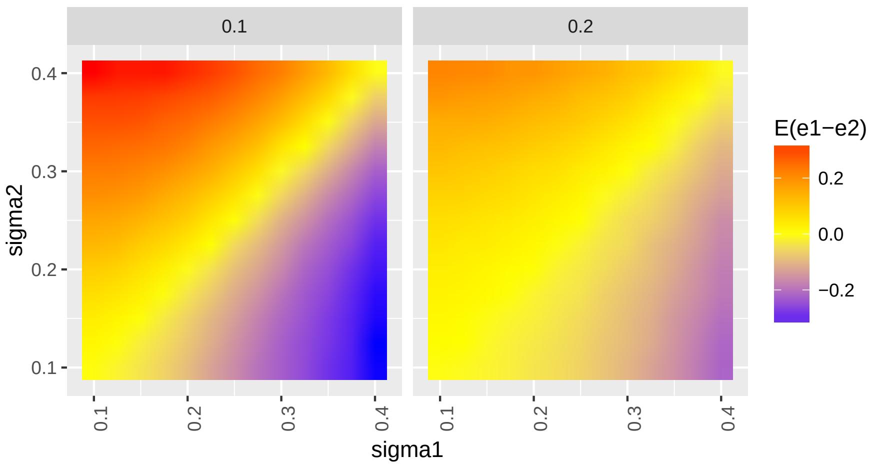


**Fig. 6.** $E(e_{a,1} - e_{a_2})$: mean difference of evaluation of company 1 and 2 (for two values 0.1 and 0.2 of $\omega$).

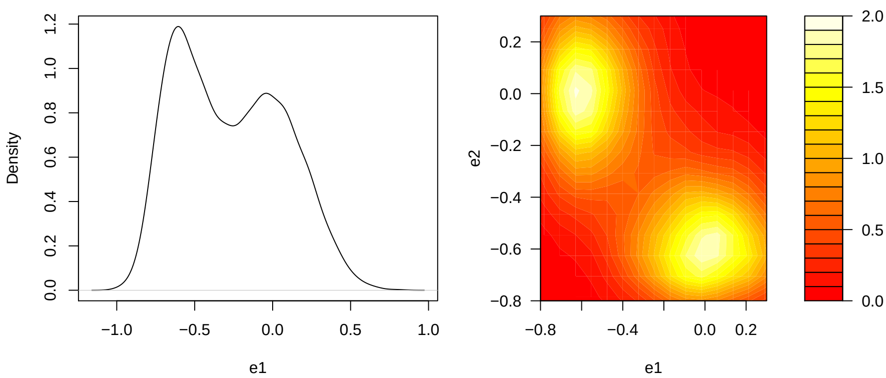


**Fig. 7.** Density of $e_{a,1}$ and joint density distribution of $(e_{a,1}, e_{a,2})$.

## 4 Results for SPA-connected agents

In this section we extend the analysis onto network effects induced by the SPA model that is a stochastic and geometric model of complex networks. Here it is used to model social connections between agents. We first start with specification of SPA model and then present analysis of model performance.

### 4.1 Spatial Preferential Attachment model

The *Spatial Preferential Attachment* (SPA) model, introduced in Aiello et al. (2009), is designed as a model for the World Wide Web and combines geometry and preferential attachment, as its name suggests. Setting the SPA model apart is the incorporation of "spheres of influence" to accomplish preferential attachment: the greater the degree of a vertex, the larger its sphere of influence, and hence the higher the likelihood of the vertex gaining more neighbours.

We now give a precise description of the SPA model. Let $S = [0,1]^m$ be the unit hypercube in $\mathbb{R}^m$, equipped with the torus metric derived from any of the $L_k$ norms. This means that for any two points $x$ and $y$ in $S$,

$$d(x,y) = \min\left\{||x-y+u||_k \,:\, u \in \{-1,0,1\}^m\right\}.$$

The torus metric thus "wraps around" the boundaries of the unit square; this metric was chosen to eliminate boundary effects. The parameters of the model consist of the *link probability* $p \in [0,1]$, and two positive constants $A_1$ and $A_2$, which, in order to avoid the resulting graph becoming too dense, must be chosen so that $pA_1 < 1$. The SPA model generates stochastic sequences of directed graphs $(G_t : t \geq 0)$, where $G_t = (V_t, E_t)$, and $V_t \subseteq S$. Let $\deg^-(v,t)$ be the in-degree of the vertex $v$ in $G_t$, and $\deg^+(v,t)$ its out-degree. We define the *sphere of influence* $S(v,t)$ of the vertex $v$ at time $t \geq 1$ to be the ball centered at $v$ with volume $|S(v,t)|$ defined as follows:

$$|S(v,t)| = \min\left\{\frac{A_1 \deg^-(v,t) + A_2}{t}, 1\right\}. \tag{1}$$

The process begins at $t = 0$, with $G_0$ being the null graph. Time step $t$, $t \geq 1$, is defined to be the transition between $G_{t-1}$ and $G_t$. At the beginning of each time step $t$, a new vertex $v_t$ is chosen *uniformly at random* from $S$, and added to $V_{t-1}$ to create $V_t$. Next, independently, for each vertex $u \in V_{t-1}$ such that $v_t \in S(u, t-1)$, a directed link $(v_t, u)$ is created with probability $p$. Thus, the probability that a link $(v_t, u)$ is added in time-step $t$ equals $p\,|S(u,t-1)|$.

The SPA model produces scale-free networks, which exhibit many of the characteristics of real-life networks (see Aiello et al. 2009; Cooper et al. 2014). In Janssen et al. (2013a), it was shown that the SPA model gave the best fit, in terms of graph structure, for a series of social networks derived from Facebook. In Janssen et al. (2013b), some properties of common neighbours were used to explore the underlying geometry of the SPA model and quantify vertex similarity based on distance in the space. However, the distribution of vertices in space was assumed to be uniform Janssen et al. (2013b) and so in Janssen et al. (2016) non-uniform distributions were investigated which is clearly a more realistic setting. Finally, in Ostroumova Prokhorenkova et al. (2017) modularity of this model was investigated, which is a global criterion to define communities and a way to measure the presence of community structure in a network.

Specifically, in Aiello et al. (2009) (Theorem 1.1) it was proved that the SPA model generates a graph with a power law in-degree distribution with exponent $1 + 1/(pA_1)$. On the other hand, the average out-degree is asymptotic to $pA_2/(1-pA_1)$ (see Theorem 1.3 in Aiello et al. 2009). In this text we take $m = 2$, $k = 2$ (two-dimensional Euclidean space), and a graph of $|A| = 10,000$ agents, $A_1 = 1$, $A_2 = 6$ and $p = 0.5$. This means that in our simulation power law has coefficient 3 and the average out-degree (and so also in-degree) is 6.

### 4.2 Results: finite state-space model

The model is still Markov process, however its state space has now size $9^{|A|}$. Moreover, the process is still irreducible and positive recurrent given the assumed domains of parameters. Therefore, it has a unique steady state $\pi$.

**Table 1.** Influence of variables on target characteristics approximated by linear regression; experiment setup: $\gamma \in \{0.25, 0.5\}$, $\delta \in \{0, 0.25, 0.5\}$, $\beta \in \{0.6, 0.7, 0.8, 0.9\}$ and $p_i(-1) = p_i(1) \in \{0.1, 0.2, 0.3, 0.4\}$. Estimates significant at 0.001 marked with *.

| variable | mean | $p_1(-1) = p_1(1)$ | $p_2(-1) = p_2(1)$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|
| degcor1 | 0.0583 | 0.0888* | -0.0875* | 0.2584* | -0.0036 | 0.1723* |
| $E(e_{a,1})$ | -0.1346 | -0.4789* | 0.2230* | -0.5920* | -0.0495 | 0.2300* |
| $\Pr(e_{a,1} > e_{a,2})$ | 0.3219 | -0.1408* | 0.5299* | 0.0735* | 0.0035 | -0.0201 |
| $\Pr(e_{a,1} = e_{a,2})$ | 0.3566 | -0.3869* | -0.3878* | -0.1352* | -0.0050 | 0.0450 |
| edgecor1 | 0.0594 | 0.0151 | -0.0131 | 0.0245* | 0.0712* | 0.2064* |

In the following analysis by degcor1 we denote correlation of in-degree of agent $a$ and her $e_{a,1}$ and by edgecor1 we denote correlation of $e_{a,1}$ and $e_{b,1}$ for all agents $a$ and $b$ that are connected by an edge.

In Table 1 we concentrate our analysis on means and want to understand the influence of parameter $\delta$ on the results. The analyzed data were collected from 384 runs of the simulation for Cartesian product of $p_1, p_2 \in \{0.1, 0.2, 0.3, 0.4\}$, $\beta \in \{0.6, 0.7, 0.8, 0.9\}$, $\gamma \in \{0.25, 0.5\}$ and $\delta \in \{0, 0.25, 0.5\}$. We report the parameters of the influence of input variables on simulation outputs estimated using linear regression metamodel.

On the average, agents with higher in-degrees have higher evaluations of product qualities. Remembering that it is on the average negative it means that higher in-degree reduces bias in evaluation of product quality. Also we observe that agents that are connected by edge have on the average positive correlation of opinions. The analysis of parameters at $\delta$ variable shows that it has relatively low impact of the structure of preferences in the population ($\Pr(e_{a,1} > e_{a,2})$ and $\Pr(e_{a,1} = e_{a,2})$ variables) — the structure of bimodality is approximately similar to no-network case. However, higher values of $\delta$ strongly reduce bias of estimates ($E(e_{a,1})$) and increase correlation of opinion with degree and between agents that are connected in the network.

### 4.3 Results: continuous state-space model

**Table 2.** Influence of variables on target characteristics approximated by linear regression; experiment setup: $\lambda_1 \in \{0.25, 0.5\}$, $\lambda_2 \in \{0, 0.25, 0.5\}$, $\omega, \sigma_1, \sigma_2 \in \{0.1, 0.2, 0.3, 0.4\}$. Estimates significant at 0.001 marked with *.

| variable | mean | $\sigma_1$ | $\sigma_2$ | $\omega$ | $\lambda_1$ | $\lambda_2$ |
|---|---|---|---|---|---|---|
| degcor1 | 0.0731* | 0.3940* | -0.1352* | -0.3431* | 0.1650* | 0.2022* |
| $E(e_{a,1})$ | -0.0458* | -0.3502* | 0.0873* | 0.2460* | -0.1357* | 0.0536* |
| $\Pr(e_{a,1} > e_{a,2})$ | 0.5010* | -0.6479* | 0.6559 | 0.0064 | -0.0045 | 0.0038 |
| edgecor1 | 0.2027 | 0.0190 | -0.0187 | -0.0353 | 0.0620 | 0.5657* |

The results are analogous to the finite state-space case. In Table 2 we still concentrate our analysis on means but this time we want to observe the influence of parameter $\lambda_2$ on the results. The analyzed data were collected from 384 runs of the simulation for Cartesian product of $\sigma_1, \sigma_2 \in \{0.1, 0.2, 0.3, 0.4\}$, $\omega \in \{0.6, 0.7, 0.8, 0.9\}$, $\lambda_1 \in \{0.25, 0.5\}$ and $\lambda_2 \in \{0, 0.25, 0.5\}$. We report the parameters of the influence of input variables on simulation outputs estimated using linear regression metamodel.

On the average, agents with higher in-degree have higher evaluation of product quality. Remembering that it is on the average negative it means that higher in-degree reduces bias in evaluation of product quality. Also we observe that agents that are connected by edge have on the average positive correlation of opinions. The analysis of parameters at $\lambda_2$ variable shows that it has relatively low impact of the structure of preferences in the population ($\Pr(e_{a,1} > e_{a,2})$ variable) — the structure of bimodality is approximately similar to no-network case. However, higher $\lambda_2$ reduces bias of estimates ($E(e_{a,1})$) and increases correlation of opinion with degree and between agents that have a connection in a graph.

## 5 Concluding remarks

Influence of quality uncertainty on markets has been traditionally studied in context of asymmetric information and rational expectations, an approach rooted in the seminal publication of Akerlof (1970). In this setting, uncertain quality causes a market to degenerate or even vanish altogether. Izquierdo and Izquierdo (2007) have demonstrated that equivalent results are obtained on markets without an a priori assumption of information asymmetry but with consumers having adaptive expectations of quality. In this paper we contributed to this stream of research by proposing a model of a market with multiple suppliers and consumers adaptively switching suppliers to maximize satisfaction. We made several interesting observations.

First, we noticed that under assumption of adaptive expectations quality uncertainty is a sufficient condition for endogenous differentiation of consumer preferences towards suppliers. As depicted on the joint density plot on Fig.7 this effect takes shape of a bimodal distribution of expected quality with majority of consumers having clear preference for one of the suppliers and almost none being indifferent. Interestingly the effect is observed regardless if network effects are taken into account or not. Traditionally in economic modelling this kind of horizontal differentiation of preferences is attributed to exogenous factors such as purposeful diversification of product characteristics by sellers to increase their market power. As we have shown similar effects may occur spontaneously and without intentional effort, by means of random variations of a product quality.

Next, we found out that lowering quality variability provides a competitive advantage to suppliers, as those who second-order dominate quality distribution of the competitors, systematically increase their market share to eventually take over the market (Fig.4). Note that we did not assume consumers to be risk averse so this effect emerged as the endogenous property of the model. This finding may have important practical implications as it provides credible justification for implementing quality assurance policies such as TQM or Six Sigma which are sometimes criticised for being merely costly "fads" having no theoretical underpinning (Linderman et al. 2003; Schroeder et al. 2008).

Finally by embedding consumers in a socially realistic SPA network we have the following findings: (a) agents with higher in-degree are better informed, (b) there is a correlation of beliefs of agents that are connected in a network, (c) higher rate of learning from neighbours reduces average bias of expectations. The analysis of dynamics of this process shows that the time to reach steady-state in the model is dramatically accelerated by social interaction as more signals are reaching the customers per one time period. This discovery reinforces the above practical conclusions regarding quality assurance policies as it indicates that the observed effects may strongly influence real markets, where information about bad quality product can spread fast and be hard to erase later. This effect is explained by our model and has been witnessed many times by companies in social media like Facebook or Twitter.

In next steps we will test robustness of our results under more “rational” Bayesian quality estimators. Another interesting question to be addressed is if lower quality could be compensated by its lower volatility i.e. is there a trade-off between expected quality and its variance. Numerical experiments confirm such possibility but this remains to be proven.

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