Common adversaries form alliances: modelling complex networks via anti-transitivity[⋆]

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Abstract. Anti-transitivity captures the notion that enemies of enemies are friends, and arises naturally in the study of adversaries in social networks and in the study of conflicting nation states or organizations. We present a simplified, evolutionary model for anti-transitivity influencing link formation in complex networks, and analyze the model's network dynamics. The Iterated Local Anti-Transitivity (or ILAT) model creates anti-clone nodes in each time-step, and joins anti-clones to the parent node's non-neighbor set. The graphs generated by ILAT exhibit familiar properties of complex networks such as densification, short distances (bounded by absolute constants), and bad spectral expansion. We determine the cop and domination number for graphs generated by ILAT, and finish with an analysis of their clustering coefficients. We interpret these results within the context of real-world complex networks and present open problems.

1 Introduction

Transitivity is a pervasive and folkloric notion in social networks, summarized in the adage that "friends of friends are more likely friends". A simplified, deterministic model for transitivity was posed in [3,4], where nodes are added over time, and each node's *clone* is adjacent to it and all of its neighbors. The resulting Iterated Local Transitivity (or ILT) model, while elementary to define, simulates many properties of social and other complex networks. For example, as shown in [4], graphs generated by the model densify over time, have the small world property (that is, small distances and high local clustering), and exhibit bad spectral expansion. For further properties of the ILT model, see [5,12]

Complex networks contain numerous mechanisms governing link formation, however. Structural balance theory in social network analysis cites several mechanisms to complete triads [11]. Another folkloric adage is that "enemies of enemies are more likely friends". Adversarial relationships may be modelled by non-adjacency, and so we have the resulting closure of the triad as described in Figure 1.
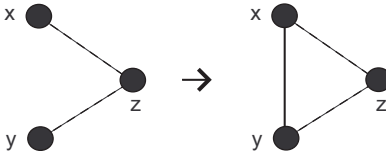

Fig. 1. Nodes x and y share z as a mutual adversary, and so form an alliance.

Such triad closure is suggestive of an analysis of adversarial relationships between nodes as one mechanism for link formation. For instance, in social networks, we may consider both friendship ties and enmity (or rivalry) between actors. We may also consider opposing networks of nation states or rival organizations, and consider alliances formed by mutually shared adversaries. See [10] for a recent study using the spatial location of cities to form an interaction network, where links enable the flow of cultural influence, and may be used to predict the rise of conflicts and violence. Another example comes from market graphs, where the nodes are stocks, and stocks are adjacent as a function of their correlation measured by a threshold value $\theta \in (0, 1)$. Market graphs were considered in the case of negatively correlated (or adversarial) stocks, where stocks are adjacent if $\theta < \alpha$, for some positive α; see [1].

In the present paper, we consider a simplified, deterministic model for anti-transitivity in complex networks. The Iterated Local Anti-Transitivity (or ILAT) model duplicates nodes in each time-step by forming *anti-clone* nodes, and joins them to the parent node's non-neighbor set. We give a precise definition of the model below in the next section. Perhaps unexpectedly, graphs generated by the ILAT model exhibit familiar properties of complex networks such as den-

sification, small world properties, and bad spectral expansion (analogously to, but different from properties exhibited by ILT).

We organize the discussion in this extended abstract as follows. In Section 2, we give a precise definition of the ILAT model and examine its basic properties. We prove that graphs generated by ILAT densify over time. We derive the density of ILAT graphs, and consider their degree distribution. In Section 3, we prove that ILAT graphs have diameter 3 for sufficiently large time-steps (regardless of the initial graph). Further, we determine after several time-steps, ILAT graphs have cop number 2 and domination number 3. We include in Section 4 an analysis of the clustering coefficients and provide upper and lower bounds. The final section interprets our results within real-world complex networks, and presents open problems derived from the analysis of the model.

We consider undirected graphs throughout the paper. For background on graph theory, the reader is directed to [13]. Additional background on complex networks may be found in the book [2].

2 The ILAT model

The Iterated Local Anti-Transitivity (or ILAT) model generates a sequence $(G_t : t \geq 0)$ of graphs over a sequence of discrete time-steps. The one parameter of the model is the initial graph G_0. Assuming the graph at time G_t is defined, we define G_{t+1} as follows. For a given node $x \in V(G_t)$, define its *anti-clone* x' as a new node adjacent to non-neighbors of x. More precisely, x' is adjacent to all nodes in $N^c(x)$, where $N^c(x) = \{y \in V(G_t) : xy \notin E(G)\}$. To form G_{t+1}, to each node x add its anti-clone x'.

The intuition behind that model is that the anti-clone x' is adversarial with x, and non-neighbors of x (that is, its own adversaries) become allied with x'. This process, therefore, iteratively applies the triad closure in Figure 1. Note that the number of nodes doubles in each time-step, and the set of anti-clones forms an independent set. See Figure 2 for an example.

We introduce some simplifying notation. Let n_t be the number of nodes at time t, e_t be the number of edges at time t, and the degree of a node x at time t will be denoted $\deg_t(x)$. We define the *co-degree* of x at time t as $\deg^c_t(x) = n_t - \deg_t(x) - 1$. It is straightforward to
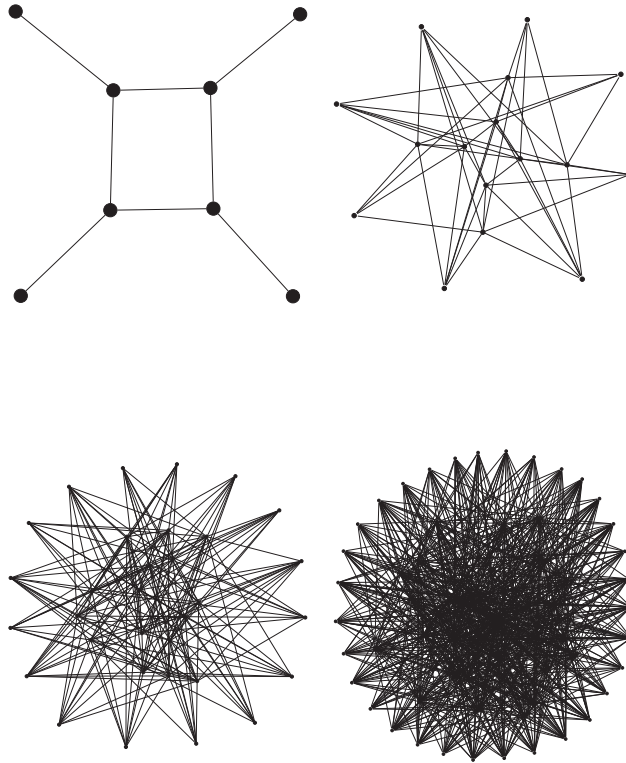
Fig. 2. An example of the first four time-steps of the ILAT model, where the initial graph is the four-cycle C_4.

note that for $t \geq 1$, $n_t = 2n_{t-1} = 2^t n_0$. Further, for an existing node $x \in V(G_t)$,

$$\deg_{t+1}(x) = n_t - 1 \tag{1}$$

$$\deg_{t+1}(x') = \deg^c_t(x). \tag{2}$$

The ILAT model generates graphs that densify as we prove next. While the proof is elementary, the result is not a priori obvious from the model. One interpretation is that in networks where anti-transitivity is pervasive, we expect that many alliances form in the network over time.

Theorem 1. *The ratio e_t/n_t tends to infinity with t.*

Proof. Note that by the definition of the model and (2), we have that

$$e_{t+1} = e_t + \sum_{x \in V(G_t)} \deg_t{}^c(x)$$

$$= e_t + n_t{}^2 - 2e_t - n_t$$

$$= n_t{}^2 - e_t - n_t.$$

Solving this recurrence, we derive that

$$e_t = n_{t-1}{}^2 \left(\frac{4}{5}\right) \left(1 - \left(-\frac{1}{4}^{t-1}\right)\right) - n_{t-1} \left(\frac{2}{3}\right) \left(1 - \left(-\frac{1}{2}^{t-1}\right)\right)$$

$$= 2^{2t}(n_0)^2 \left(\frac{1}{5}\right) \left(1 - \left(-\frac{1}{4}^{t-1}\right)\right) (1 - o(1)).$$

Hence, we obtain that $e_t/n_t = \Omega(2^t)$. $\qquad\qquad\square$

Note that Theorem 1 immediately gives the limiting density of ILAT graphs. Let D_t be the density of G_t; that is, $D_t = \frac{e_t}{\binom{n_t}{2}}$.

Corollary 1. *As $t \to \infty$, we have that $D_t \to 2/5$.*

We next consider the degrees of vertices in the graph G_t. For each node x at time t, we create its anti-clone x' at time $t+1$. Then at time $t+2$ we create x'' from x and $(x')'$ from x'. For any node x that was created at a time-step $k < t$, we have directly from (1) that

$$\deg_t(x) = \frac{n_t}{2} - 1.$$

If $t > 1$, then of the newly created nodes, half are anti-clones x' of nodes x that have already existed at time $t - 2$, and therefore, their degree at time $t - 1$ was

$$\deg_{t-1}(x) = \frac{n_{t-1}}{2} - 2 = \frac{n_t}{4} - 1.$$

These anti-clones have at time t,

$$\deg_t(x') = n_{t-1} - \deg_{t-1}(x) = \frac{n_t}{4} + 2.$$

Similarly, if $t > 2$ then there are $\frac{n_t}{8}$ nodes y'' created at time t that are anti-clones of nodes y' created at time $t - 1$ from nodes y at least as old as $t - 3$. Then since by the previous argument $\deg_{t-1}(y') = \frac{n_{t-1}}{4} + 2$, we have that

$$\deg_t(y'') = \frac{3n_t}{8} - 1.$$

If we continue in this fashion, then by induction we will find that at time t, we have that $2^{-k}n_t$ nodes have degree $a_k + (-1)^{k-1}2$ provided that for $k < t$:

$$a_1 = \frac{n_t}{2} - 1,$$

and

$$a_k = \frac{1}{2} - \frac{a_{k-1}}{2}.$$

If $t > 1$, then of the newly created nodes, half are anti-clones x' of nodes x that already existed at time $t - 2$. Therefore, the degree of those nodes x at time $t - 1$ was

$$\deg_{t-1}(x) = \frac{n_{t-1}}{2} - 1 = \frac{n_t}{4} - 1.$$

Their new anti-clones x' have, at time t,

$$\deg_t(x') = n_{t-1} - \deg_{t-1}(x) = \frac{n_t}{4} + 1.$$

Similarly, if $t > 2$ then there are $\frac{n_t}{8}$ nodes y'' created at time t that are anti-clones of nodes y' created at time $t - 1$ from nodes y at least as old as $t - 3$. Then since by the previous argument $\deg_{t-1}(y') = \frac{n_{t-1}}{4} + 1$, we have that

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$$a_1 = \frac{n_t}{2} - 2,$$

and
$$a_k = \frac{1}{2} - \frac{a_{k-1}}{2}.$$

3 Distances and graph parameters

The distances within graphs generated by ILAT become very small, with diameter 3. Hence, highly anti-transitive networks exhibit short paths between nodes; this occurs at time-step $t = 2$, regardless of the starting diameter of G_0.

Theorem 2. *Let $t \geq 2$, then the diameter* $\mathrm{diam}(G_t)$ *of G_t is 3.*

Note that the value $t = 2$ in Theorem 2 is sharp. For example, we may take G_0 to be a path of length 4. Or we may consider an initial graph of K_3, in which case the graph at $t = 1$ is disconnected.

Proof of Theorem 2. We show first that for $t \geq 1$, the diameter of G_t is at least 3. To see this, consider the distance between some node x that existed at time $t - 1$ and its anti-clone x' created at time t. They are not adjacent and have no common neighbors, and so we have that $d(x, x') \geq 3$.

We next show that for $t \geq 2$, any two nodes that are not newly created are at most distance 2 apart. For this, let x, y be two distinct nodes that already existed at time $t - 1$. Since the node degree at time $t - 1$ is bounded by $n_t/4 - 1$, by the pigeonhole principle there is another node z that also existed at $t - 1$ that is not adjacent to either of them. Hence, z' is adjacent to both nodes and so $d(x, y) \leq 2$.

Let x', y' be two separate nodes newly anti-cloned from some nodes x, y. Since the node degree at time $t - 1$ is bounded by $\max\{0, n_t/4 - 1\}$, by the pigeonhole principle there is another node z that also existed at $t - 1$ that is not adjacent to either x or y. Then z is adjacent to both x' and y', and so $d(x', y') \leq 2$. Hence, any two nodes that both newly created are at most distance 2 apart.

The only case we have not considered are pairs of nodes where one is newly created and one is not. But if $t \geq 3$, then every newly created node has a neighbor that is not newly created and vice versa. Therefore, any such pair can be connected by a path of length at most 3. □

The pairs of nodes we have not considered so far are ones where exactly one node is newly created, but is not a anti-clone of the other. If they are not adjacent, then we would like to know if they have a common neighbor. Let the node that already existed at time $t-1$ be x, and the newly created node be y', cloned from some node $y \neq x$. Nodes x and y' can have a common neighbor unless the neighborhood of x at time $t-1$ (other than possibly y itself) was a subset of the neighborhood of y at time $t-1$ (which would be the case when $x = y$).

Theorem 3. *If x and y are nodes of G_t that are not newly created at time t, with $t \geq 2$ and $x \neq y$, and it is not the case that both x and y belonged to G_0, then $d(x, y') \leq 2$.*

Proof. Unless x and y are adjacent, we have that $d(x, y') = 1$. So suppose that x and y are adjacent. Suppose that they did not both belong to the initial graph G_0. Since they are adjacent, one of them was created later than the other. If y was created later, then every neighbor of x that was created at the same time as y is now a common neighbor of x and y'. If x was created later, but before $t-1$, then every node adjacent to y but not x at the time produced a anti-clone of the type we need. We are left with a case where x was created at time $t-1$, and y was created earlier.

We want to find a common neighbor of x and y' that was created at $t-2$ or earlier. x was created at time $t-1$, so it was cloned from a node with has either $n_t/8 - 2$, $n_t/16 + 2$ or about $n_t/12$ neighbors that already existed at time $t-1$, and so x has either $n_t/8 + 2$, $3n_t/16 - 2$, or about $n_t/6$ neighbors older than itself. By the same argument, y' has either $n_t/8 + 2$, $3n_t/16 - 2$, or about $n_t/6$ neighbors at least as old as $t-2$. There are in total $n_t/4$ nodes at least as old as $t-2$. So by the pigeonhole principle, they must have such a neighbor in common. $\square$

Let L_t denote the average distance at time t.

Corollary 2. *The average distance L_t tends to 1.6 in t.*

Proof. Notice that the number of pairs such that both x and y belong to G_0 is negligible, so will not change the average distance limit. Of

the remaining pairs of vertices, a proportion of 0.4 are adjacent and the rest are at distance 2. We can thus, conclude that

$$\lim_{t \to \infty} L_t = 1.6.$$

$\square$

We next turn to a brief discussion of the domination and cop numbers of the ILAT graphs. As we have noticed with other parameters such as the diameter and average distance, these two parameters are bounded above by very small constants. For more on these graph parameters, see [6] (we omit their definitions here as they are well-known and owing to space constraints). As a possible interpretation of these, we note that in networks exhibiting high anti-transitivity, a few important nodes emerge (either dominating nodes, or mobile agents represented by cops) which can reach all other nodes. Such so-called *superpower* nodes organically emerge as important actors in the network.

Theorem 4. *In G_t such that $t \geq 3$, the domination number is 3.*

Proof. Let $A = \{x, x', (x')'\}$ be as follows. For any $1 \leq k \leq t - 1$, let x be a node that existed at time $k - 1$ and x' be the time-k anti-clone of x. Let x'' be the time-$(k+1)$ anti-clone of x'. Then any node of G_t not in A is either adjacent to x', adjacent to x'', or a node created at time $k + 1$ that is not adjacent to x', in which case it must be adjacent to x. Therefore, A is a dominating set of G_t.

If $t \geq 1$, then we can never find a dominating set of size 2. The node degrees are bounded by $\frac{n_t}{2} - 2$. Therefore, the union of neighborhoods of any two nodes contains at most $n_t - 4$ nodes. $\square$

Theorem 5. *If $t \geq 2$, then the cop number of G_t is 2.*

Proof. In a simple, omitted argument, if $t \geq 2$, the cop number of G_t is never 1. We now describe how two cops may capture the robber. Fix $v \in V(G_{t-1})$. Then each vertex of G_{t-1} is adjacent to one of v or v'. Place the cops on v and v'. Hence, the robber must begin on an anti-clone say u' newly created at time t not adjacent to either v or v'. Now there must be an x in G_t joined to u', otherwise, u is a universal vertex in G_{t-1} which is a contradiction (here is where

we use $t \geq 2$). It is straightforward to show that there is a perfect matching between x, x' and v, v', and so the cops move to x, x'. The robber must move to a vertex z in G_{t-1}. But z is joined to one of x or x' and the robber is caught in the next move. $\square$

Note that we must have $t \geq 2$ in Theorem 5 or the cop number could be larger than 2. For example, if G_0 is a K_3, then G_1 is the disjoint union of K_3 and $\overline{K_3}$, which has cop number 4.

4   Clustering coefficient

For a node v, define $c_t(v)$ to be the (local) clustering coefficient of the node v at time t. We note that in the ILAT model, older nodes exhibit significant local clustering over time.

Theorem 6. *Let $k \in \mathbb{N}$. For node v created at time k, with $t > k$, if $\lim_{t \to \infty} c_t(v)$ exists, then we have that*

$$\lim_{t \to \infty} c_t(v) = 0.4.$$

Hence, the clustering coefficient of a node v tends to 0.4 as v grows old, which matches the density of the graph.

Proof of Theorem 6. Let $c'_t(v) = c'_t$ be the density of v's non-neighborhood set at time t, and let $c''_t(v) = c''_t$ be the density between the neighborhood and the non-neighborhood of v. Hence, the number of edges with both endpoints in the neighborhood of v is $c_t(v)\binom{\deg_t(v)}{2}$, the number of edges with both endpoints in the non-neighborhood of v is $c'_t\binom{n_t - \deg_t(v) - 1}{2}$, the number of edges with one endpoint in the neighborhood of v, and the remaining number of edges in the non-neighborhood of v is $c''_t \deg_t(v)(n_t - \deg_t(v) - 1)$.

We write $a \sim b$ if $a = b(1 + o(1))$. For large t, we may approximate the degree by $\deg_t(v) \sim n_t - \deg_t(v) - 1 \sim \frac{n_t}{2}$. Further, since the total number of edges in the graph tends to $0.4\binom{n_t}{2}$, we have that

$$\frac{c_t + c'_t t + 2c''_t}{4} \sim \frac{2}{5},$$

and

$$c'_t \sim \frac{8}{5} - c_t - 2c''_t.$$

Then we may determine $c_{t+1}(v) = c_{t+1}$ by counting the edges with both endpoints in the neighborhood of v at time $t+1$. These are either the same edges that contributed to $c_t(v)$, or edges between the t-time neighborhood of v and the anti-clones of its non-neighborhood, giving the following equations:

$$c_{t+1}\binom{n_t}{2} \sim c_t\binom{n_t/2}{2} + (1 - c''_t)\frac{n_t^2}{4},$$

$$c_{t+1} \sim \frac{c_t}{4} + \frac{1 - c''_t}{2}.$$

Further, we have that

$$c''_{t+1} = \frac{c''_t}{4} + \frac{1 - c'_t}{4} + \frac{1 - c_t}{4}$$

$$c''_{t+1} = \frac{c''_t}{4} + \frac{1 - \frac{2}{5} + c_t(v) + 2c''_t}{4} + \frac{1 - c_t}{4}, \text{ and}$$

$$c''_{t+1} = \frac{3c''_t + \frac{2}{5}}{4}.$$

By hypothesis, the limiting value of c_t exists and we call this quantity c. In particular, we have that for a sufficiently large t that, $c_t(v) \sim c_{t+1} \sim c_{t+1} \sim c$. We have that

$$c_{t+2} = \frac{c_{t+1}}{4} + \frac{1 - c''_{t+1}}{2} = \frac{c_{t+1}}{4} + \frac{3}{4}\frac{1 - c''_t}{2} + \frac{1 - \frac{2}{5}}{8},$$

and so $c_{t+2} = c_{t+1} - \frac{3c_t}{16} + \frac{3}{40}$. By taking the limit as $t \to \infty$, we have that $\frac{3}{16}c = \frac{3}{40}$, and the result follows. $\qquad\square$

An open problem remains to prove that the limiting value of c_t exist. Further, computing the value of the clustering coefficient of G_t remains open.

5 Spectral expansion

For a graph $G = (V, E)$ and sets of nodes $X, Y \subseteq V$, define $E(X, Y)$ to be the set of edges in G with one endpoint in X and the other

in Y. For simplicity, we write $E(X) = E(X, X)$. The normalized Laplacian of a graph relates to important graph properties; see [7] for a reference. Let A denote the adjacency matrix and D denote the diagonal degree matrix of a graph G. Then the normalized Laplacian of G is $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$. Let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2$ denote the eigenvalues of $\mathcal{L}$. The *spectral gap* of the normalized Laplacian is defined as

$$\lambda = \max\{|\lambda_1 - 1|, |\lambda_{n-1} - 1|\}.$$

A spectral gap bounded away from zero is an indication of bad expansion properties, which is characteristic for social networks; see [9]. The next theorem represents a drastic departure from the good expansion found in binomial random graphs, where $\lambda = o(1)$; see [7,8].

Theorem 7. *If λ_t is the spectral gap of G_t, then $\lambda_t \geq 3/5 + o(1)$.*

To prove Theorem 7, we use the expander mixing lemma for the normalized Laplacian (see [7] for its proof). For sets of nodes X and Y we use the notation $\mathrm{vol}(X) = \sum_{v \in X} \deg(v)$ for the volume of X, $\bar{X} = V \setminus X$ for the complement of X, and, $e(X, Y)$ for the number of edges with one end in each of X and Y. (Note that $X \cap Y$ does not have to be empty; in general, $e(X, Y)$ is defined to be the number of edges between $X \setminus Y$ to Y plus twice the number of edges that contain only nodes of $X \cap Y$. In particular, $e(X, X) = 2|E(X)|$.)

Lemma 1. *For all sets $X \subseteq V(G_t)$,*

$$\left| e(X, X) - \frac{(\mathrm{vol}(X))^2}{\mathrm{vol}(G_t)} \right| \leq \lambda_t \frac{\mathrm{vol}(X)\mathrm{vol}(\bar{X})}{\mathrm{vol}(G_t)}.$$

Proof of Theorem 7. Let X be the set of $n_t/2$ the youngest nodes. Since X induces an independent set, we note that $e(X, X) = 0$. We derive that

$$\begin{aligned}
\mathrm{vol}(G_t) &\sim 2{n_t}^2/5, \\
\mathrm{vol}(\bar{X}) &\sim {n_t}^2/4, \quad \text{and} \\
\mathrm{vol}(X) &= \mathrm{vol}(G_t) - \mathrm{vol}(\bar{X}) \sim 3{n_t}^2/20,
\end{aligned}$$

where the second expression holds as $(n_t/2)$-many of the oldest nodes have degree $\sim n_t/2$. Hence, by Lemma 1, we have that

$$\lambda_t \geq \frac{(\mathrm{vol}(X))^2}{\mathrm{vol}(G_t)} \cdot \frac{\mathrm{vol}(G_t)}{\mathrm{vol}(X)\mathrm{vol}(\bar{X})} = \frac{\mathrm{vol}(X)}{\mathrm{vol}(\bar{X})} \sim 3/5,$$

and the proof follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6 Discussion and future work

We introduced the Iterated Local Anti-Transitivity (ILAT) model for complex networks and analyzed properties of the graphs it generates. We proved that graphs generated by ILAT densify over time, have diameter 3, and have density tending to 0.4. ILAT graphs have small dominating sets and low cop number. We analyzed the clustering coefficient of ILAT graphs, and noted that while older nodes show high (local) clustering, the (global) clustering coefficient is less than what is expected in binomial random graphs with the same expected degree. In addition, we showed that graphs generated by ILAT exhibit bad spectral expansion as found in social networks.

Theoretical results presented here for the ILAT model are suggestive of several emergent properties in networks where anti-transitivity governs link formation. For instance, the presence of small (3-element) dominating sets suggest the emergence of nodes we describe as *superpowers*, which have broad influence in the network. Such nodes may emerge naturally in real-world networks which are highly anti-transitive, owing to a high number of alliances against common adversaries. Similarly, the presence of short paths, high density, and high (local) clustering of older nodes in ILAT graphs suggests that networks, where common adversaries forge alliances, naturally form tight-knit communities that are well-connected. In the sequel, it would be interesting to empirically test these hypotheses with real-world networked data.

Besides applications of the ILAT model, it raises a number of interesting graph-theoretic questions. An open problem remains to compute the clustering coefficient for ILAT graphs. Another question is to determine the induced subgraph structure of such graphs. A characterization of the induced subgraphs of ILAT graphs (that is,

to determine its *age*) remains open. For example, do all finite trees appear as induced subgraphs of ILAT graphs?

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