INJECTIVE COLOURING OF BINOMIAL RANDOM GRAPHS

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ABSTRACT. The injective chromatic number of a graph G is the minimum number of colours needed to colour the vertices of G so that two vertices with a common neighbour receive distinct colours. In this paper, we investigate the injective chromatic number of the binomial random graph $\mathcal{G}(n,p)$ for a wide range of p. We also study a natural generalization of this graph parameter for the same model of random graphs.

1. INTRODUCTION

A *proper colouring* of a graph is a labelling of its vertices with colours such that no two vertices sharing the same edge have the same colour. A proper colouring using at most k colours is called a *proper k-colouring*. The smallest number of colours needed to properly colour a graph G is called its *chromatic number* and denoted by $\chi(G)$.

In this paper we are concerned with another notion of colouring. A set $R \subseteq V$ in a coloured graph is said to be *rainbow coloured* if every vertex in R has a different colour. An *injective colouring* of a graph is a colouring of the vertices so that the (open) neighbourhood of any vertex is rainbow. The smallest number of colours needed to colour G (in this new model) is called its *injective chromatic number*, and it is denoted by $\chi_i(G)$. This graph parameter was introduced in [8] and has been studied since then in many different deterministic contexts [2, 3, 4, 5]. In this paper we extend our knowledge of $\chi_i(G)$ to binomial random graphs.

An injective colouring need not be a proper colouring; indeed, the injective chromatic number of the Petersen graph is 5, and is achieved by an injective colouring in which each colour class induces K_2. It follows immediately from the definition of this graph parameter that $\chi_i(G)$ equals the chromatic number of the *common neighbour graph* G_{cn}, the graph on vertex set $V(G)$ where vertices are adjacent if they have a common neighbour in G. The *square* of a graph G is the graph G^2 with the same vertex set as G in which vertices are adjacent if their distance in G is at most 2. Since G_{cn} is a subgraph of G^2, we get immediately that

$$\Delta(G) \le \chi_i(G) = \chi(G_{cn}) \le \chi(G^2) \le \Delta(G)^2 + 1,$$

where $\Delta(G)$ is the maximum degree of G. The lower bound of $\Delta(G)$ is obvious; see Lemma 2.1 for the upper bound of $\Delta(G)^2+1$. In fact, $\chi_i(G) \ge \chi(G^2)/2$ and so $\chi(G^2)$ and $\chi_i(G)$ are within a factor of 2 (see [11] for more details). As we already mentioned, this graph parameter is well studied. For example, upper bounds for graph families with

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small maximum average degree have been obtained [6, 3, 13, 4, 5]. These bounds can be easily applied to planar graphs with large enough girth. For example, $\chi_i(G) \leq \lceil 3\Delta/2 \rceil$ whenever G does not have K_4 as a minor and has maximum degree Δ [2].

Finally, looking at the bigger picture, let us note that the chromatic number of G^2 has important applications in Steganography [7], which is the study of hiding messages in other media in such a way that no one, apart from the sender and desired receiver, suspects the presence of a message. As a result, for example, $\chi(G^2)$ has been studied extensively in the case where G is a planar graph [9, 15].

Let us recall the classic model of random graphs that we study in this paper. The *binomial random graph* $\mathcal{G}(n,p)$ is the random graph G with vertex set $[n]$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

All asymptotics throughout are as $n \to \infty$ (we emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded). We also use the notations $f \ll g$ for $f = o(g)$ and $f \gg g$ for $g = o(f)$. We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Since we aim for results that hold a.a.s., we will always assume that n is large enough. We often write $G \in \mathcal{G}(n,p)$ (or simply $\mathcal{G}(n,p)$) to denote a random graph drawn from the distribution $\mathcal{G}(n,p)$. For simplicity, we will write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$ (that is, when $f(n) = (1+o(1))g(n)$). Finally, we use $\log n$ to denote natural logarithms.

Next, we highlight our main results for the injective chromatic number and for its generalization. Here is our main result for the injective chromatic number.

Theorem 1.1. *Let $\omega = \omega(n) = o(\log n)$ be any function tending to infinity as $n \to \infty$. Then the following properties hold a.a.s. for $G \in \mathcal{G}(n,p)$.*

(i) *If $pn \geq \sqrt{2n(\log n + \omega)}$, then*
$$\chi_i(G) = \chi(G^2) = n.$$
(ii) *If $\sqrt{n} \leq pn < \sqrt{2n(\log n + \omega)}$, then*
$$\frac{(pn)^2}{10\log(pn)} < \chi_i(G) \leq \chi(G^2) \leq n.$$
(iii) *If $\log n \ll pn < \sqrt{n}$, then*
$$\frac{(pn)^2}{10\log(pn)} < \chi_i(G) \leq \chi(G^2) \leq (1+o(1))(pn)^2.$$

The proof can be found in Section 2. Part (i) follows immediately from Lemma 2.3. Parts (ii) and (iii) follow from Lemma 2.4 and Corollary 2.2.

Note that the result is quite sharp and shows that the injective chromatic number for binomial random graphs is close to the trivial upper bound of $\min\{\Delta(G)^2+1, n\}$. However, there is still a gap between the bounds in parts (ii) and (iii). Observe that the expected degrees of the vertices of both $\mathcal{G}(n,p)^2$ and $\mathcal{G}(n,p^2n)$ are asymptotic to $(pn)^2$. Therefore, one could conjecture that $\chi_i(\mathcal{G}(n,p)) = \Theta(\chi(\mathcal{G}(n,p)^2))$ should behave similarly to $\chi(\mathcal{G}(n,p^2n))$, which is a.a.s. of order $(pn)^2/\log(pn)$ [12, 14], as is the lower bound in part (iii). However, for example, the expected number of triangles attached to a given vertex is of order $(pn)^6/n$ in the $\mathcal{G}(n,p^2n)$ model, whereas it is of order $(pn)^3$ in the $\mathcal{G}(n,p)^2$ one (for graphs that are sparse enough). This shows that the structures of the two random models are quite different, and the precise behaviour of the injective chromatic number for this range of $p=p(n)$ is far from being clear.

Let us also investigate the following natural generalization. For a given natural number k, a *k-injective colouring* of a graph is a colouring of the vertices such that the (open) k-th neighbourhood of any vertex is rainbow. This time, the minimum number of colours needed to colour G is called its *k-injective chromatic number*, and it is denoted by $\chi_i^k(G)$. Clearly, $\chi_i(G)=\chi_i^1(G)$. Moreover, for any $k\in\mathbb{N}$, we have

$$\chi_i^k(G)\le\chi(G^{2k}).$$

Here is our main result for this generalization.

Theorem 1.2. *Let $k=k(n)\in\mathbb{N}\setminus\{1\}$, $\varepsilon\in(0,1/3)$, and suppose that $d=pn\gg\log n$. Then the following properties hold a.a.s. for $G\in\mathcal{G}(n,p)$.*

(i) *If $((2+\varepsilon)n\log n)^{1/(2k)}\le pn=o(n^{1/(k-1)})$, then*

$$\chi_i^k(G)=\chi(G^{2k})=n.$$

(ii) *If $n^{1/(2k)}\le pn<((2+\varepsilon)n\log n)^{1/(2k)}$, then*

$$\frac{(pn)^{2k}}{10k\log(pn)}\le\chi_i^k(G)\le\chi(G^{2k})\le n.$$

(iii) *If $\log n\ll pn<n^{1/(2k)}$, then*

$$\frac{(pn)^{2k}}{10k\log(pn)}\le\chi_i^k(G)\le\chi(G^{2k})\le(1+o(1))(pn)^{2k}.$$

Section 3 is devoted to show some expansion properties needed to obtain the result. The proof of the theorem can be found in Section 4. Part (i) follows immediately from Lemma 4.3. Parts (ii) and (iii) follow from Lemma 4.4 and Corollary 4.2.

Finally, let us mention that the assumption $pn=o(n^{1/(k-1)})$ in Part (i) is not needed to extend the result for $\chi(G^{2k})$ to denser graphs, since $\chi(G^{2k})$ is monotonic. On the other hand, it is not the case for $\chi_i^k(G)$. This graph parameter is definitely not monotonic: for example, clearly $\chi_i^k(G)=1$ when the diameter of G is smaller than k (the desired property is vacuously true as the k-th neighbourhood of any vertex is empty). Hence, the assumption $pn=o(n^{1/(k-1)})$ cannot be dropped, since the diameter of $\mathcal{G}(n,p)$ changes from k to $k-1$ at a point in which pn is of order $(n\log n)^{1/(k-1)}$, as shown in the following result.

Lemma 1.3 ([1], Corollary 10.12)**.** *Suppose that $d=p(n-1)\gg\log n$ and*

$$d^i/n-2\log n\to\infty\qquad\text{and}\qquad d^{i-1}/n-2\log n\to-\infty.$$

Then the diameter of $G\in\mathcal{G}(n,p)$ is equal to i a.a.s.

2. Injective colouring—the case $k = 1$

For a vertex $v \in V$, let $S_i(v)$ and $N_i(v)$ denote the set of vertices at distance i from v and the set of vertices at distance at most i from v, respectively. For any $V' \subseteq V$, let $S_i(V') = \bigcup_{v \in V'} S_i(v)$ and $N_i(V') = \bigcup_{v \in V'} N_i(v)$.

2.1. **Upper bound.**

Let us start with the following observation that holds for deterministic graphs before moving back to random graphs. It is a well-known upper bound (see, for example, [8]); however, we present the proof for completeness.

Lemma 2.1. *For any graph G, we have*

$$\chi(G^2) \le \Delta^2 + 1,$$

where $\Delta = \Delta(G)$ is the maximum degree of G.

Proof. Let us fix any order of vertices and consider a greedy algorithm assigning colours to vertices in such order. A colour assigned to a vertex v has to be different from colours already assigned to $N_2(v) \setminus \{v\}$. Since $|N_2(v) \setminus \{v\}| \le \Delta + \Delta(\Delta - 1) = \Delta^2$, we conclude that $\Delta^2 + 1$ colours will always suffice. □

In order to apply this observation to random graphs, we will use the following version of *Chernoff's bound*. Suppose that $X \in \mathrm{Bin}(n, p)$ is a binomial random variable with expectation $\mu = np$. If $0 < \delta < 3/2$, then

$$\Pr\left(|X - \mu| \ge \delta\mu\right) \le 2\exp\left(-\frac{\delta^2 \mu}{3}\right). \tag{1}$$

(For example, see Corollary 2.3 in [10].)

Corollary 2.2. *Suppose that $d = pn \gg \log n$. Then, a.a.s.*

$$\chi(\mathcal{G}(n,p)^2) \le (1 + o(1))(pn)^2.$$

Proof. Let $\omega = \omega(n) = pn/\log n$, and consider $G \in \mathcal{G}(n,p)$. (Since $pn \gg \log n$, note that $\omega(n) \to \infty$ as $n \to \infty$.) We will use Lemma 2.1, so it is enough to show that $\Delta = \Delta(G) \sim pn$. Let $v \in V(G)$. Since the degree of v is a random variable $\mathrm{Bin}(n-1, p)$ with expectation $p(n-1) \sim pn = \omega \log n$, we may apply Chernoff's bound (1) to get that

$$\begin{aligned}
\Pr\left(|\deg(v) - \mathbb{E}\deg(v)| \ge \mathbb{E}\deg(v)/\omega^{1/3}\right) &\le 2\exp\left(-\frac{\omega^{-2/3}\mathbb{E}\deg(v)}{3}\right) \\
&= 2\exp\left(-\frac{\omega^{1/3}\log n}{3 + o(1)}\right) = o(n^{-1}).
\end{aligned}$$

Hence, by the union bound, a.a.s. $\deg(v) \sim pn$ for all $v \in V(G)$, and the proof is finished. □

2.2. **Lower bound.** We will deal with dense and sparse random graphs independently. Let us start with the dense case.

Lemma 2.3. *Suppose that* $d = pn \ge \sqrt{2n(\log n + \omega)}$*, where* $\omega = \omega(n) = o(\log n)$ *is any function tending to infinity as* $n \to \infty$*. Then, a.a.s.*

$$\chi_i(\mathcal{G}(n,p)) = n.$$

Proof. We are going to show that a.a.s. any two vertices of $G \in \mathcal{G}(n,p)$ have a common neighbour. This will finish the proof as it implies that no two vertices can have the same colour assigned.

Let X be the random variable counting the number of pairs of vertices for which there is no common neighbour. Using the fact that $1 - x \le \exp(-x)$ for any $x \in \mathbb{R}$, we get

$$\begin{aligned} \mathbb{E}X &= \binom{n}{2}(1-p^2)^{n-2} \le n^2 \exp(-p^2(n-2)) \le n^2 \exp\left(-2(\log n + \omega)\left(1 - \frac{2}{n}\right)\right) \\ &= \exp(-2\omega + o(1)) = o(1). \end{aligned}$$

The result holds by the first moment method. □

Next, we investigate the sparse case.

Lemma 2.4. *Suppose that* $\log n \ll d = pn < \sqrt{2n(\log n + \omega)}$*, where* $\omega = \omega(n) = o(\log n)$ *is any function tending to infinity as* $n \to \infty$*. Then, a.a.s.*

$$\chi_i(\mathcal{G}(n,p)) > \frac{(pn)^2}{10\log(pn)}.$$

Proof. Let $G = (V,E) \in \mathcal{G}(n,p)$. We are going to show that a.a.s. for every set $S \subseteq V$ of cardinality $s = \lceil (10n\log(pn))/(pn)^2 \rceil$ there exists a vertex $v = v(S) \in V \setminus S$ that is adjacent to precisely two vertices of S. (Note that "precisely" could be replaced by "at least" but this version will turn out to be more convenient and the improvement would be negligible.) This will finish the proof as it implies that no colour class has size at least s and so $\chi_i(G) \ge n/(s-1) > (pn)^2/(10\log(pn))$. Indeed, if vertices of some set $S \subseteq V$ of cardinality s are of the same colour, then $v = v(S)$ would not have a rainbow neighbourhood.

Let X_s be the random variable counting the number of sets of vertices of cardinality s for which there is no vertex with precisely two neighbours in S. It follows that

$$\begin{aligned} \mathbb{E}X_s &= \binom{n}{s}\left(1 - \binom{s}{2}p^2(1-p)^{s-2}\right)^{n-s} \\ &\le \left(\frac{en}{s}\right)^s \exp\left(-\frac{s(s-1)}{2}p^2 e^{-ps(1+o(1))}(n-s)\right) \\ &\le (pn)^{2s} \exp\left(-(1+o(1))\frac{s^2}{4}p^2 n\right). \end{aligned}$$

The last inequality follows from the fact that $s \ge (10n\log(pn))/(pn)^2 \ge 5/2 + o(1) \ge 2$ (since $pn < (\sqrt{2} + o(1))\sqrt{n\log n}$) and so $s(s-1)/2 \ge s^2/4$, and the fact that $ps \le$

$(10\log(pn))/(pn) + p = o(1)$ (since $pn \gg \log n$). We get that

$$\mathbb{E}X_s \le \exp\left(s\left(2\log(pn) - (1+o(1))\frac{s}{4}p^2n\right)\right) = \exp\left(-\left(\frac{1}{2}+o(1)\right)s\log(pn)\right) = o(1).$$

The result holds by the first moment method. $\square$

3. Expansion properties

In order to deal with the generalization of the injective chromatic number we will need the following expansion-type properties of random graphs.

Lemma 3.1. *Let $\omega = \omega(n)$ be any function tending to infinity as $n \to \infty$. Suppose that $\omega \log n \le d = p(n-1) = o(n)$. Then the following properties hold a.a.s. for $G = (V, E) \in \mathcal{G}(n, p)$. For any $V' \subseteq V$ with $|V'| \le 2$ and any $i \in \mathbb{N}$ such that $d^i = o(n)$,*

$$|S_i(V')| = \left(1 + O\left(\frac{1}{\sqrt{\omega}}\right) + O\left(\frac{d^i}{n}\right)\right) d^i|V'| \sim d^i|V'|.$$

In particular, for every $x, y \in V$ $(x \neq y)$ we have

$$|S_i(x) \setminus S_i(y)| = \left(1 + O\left(\frac{1}{\sqrt{\omega}}\right) + O\left(\frac{d^i}{n}\right)\right) d^i \sim d^i.$$

Proof. We will show that a.a.s. for every $V' \subseteq V$ with $|V'| \le 2$ and $i \in \mathbb{N}$ we have the desired concentration for $|S_i(V')|$, provided that $d^i = o(n)$. The statement for any pair of vertices x, y will follow immediately (deterministically) from this. In order to investigate the expansion property of neighbourhoods, let $Z \subseteq V$, $z = |Z|$. Additionally, consider the random variable X that counts the number of vertices in $V \setminus Z$ that are adjacent to some vertex in Z, formally described as $X = X(Z) = |N_1(Z) \setminus Z|$. We will bound X in a stochastic sense. In order to achieve this, two things need to be estimated: the expected value of X, and the concentration of X around its expectation.

Note that X is distributed as $\text{Bin}\left(n - z, 1 - \left(1 - \frac{d}{n-1}\right)^z\right)$. Since for $x = o(1)$ we have $(1-x)^z = e^{-xz(1+O(x))}$ and also $e^{-x} = 1 - x + O(x^2)$, it is clear that

$$\begin{aligned}
\mathbb{E}[X] &= (n-z)\left(1 - \left(1 - \frac{d}{n-1}\right)^z\right) \\
&= (n-z)\left(1 - \exp\left(-\frac{dz}{n}(1 + O(d/n))\right)\right) \\
&= (n-z)\frac{dz}{n}\big(1 + O(d/n) + O(dz/n)\big) \\
&= dz(1 + O(dz/n)), \qquad (2)
\end{aligned}$$

provided $dz = o(n)$.

Using Chernoff's bound (1) with $\delta = 2/\sqrt{\omega}$, we get that the expected number of sets V' satisfying

$$\big|X(V') - \mathbb{E}[X(V')]\big| > \delta d|V'|$$

and $|V'| \leq 2$ is at most

$$\sum_{z\in\{1,2\}} 2n^z \exp\left(-\frac{\delta^2 zd}{3+o(1)}\right) \leq \sum_{z\in\{1,2\}} 2n^z \exp\left(-\frac{\delta^2 z\omega \log n}{3+o(1)}\right) = o(1),$$

since $d \geq \omega \log n$. Hence, a.a.s.

$$|N_1(V') \setminus V'| = d|V'|(1 + O(1/\sqrt{\omega}) + O(d/n)), \tag{3}$$

and the statement holds for $i = 1$ a.a.s. (since $S_1(V')$ and $N_1(V')\setminus V'$ differ by at most 2 vertices, deterministically). Now, we will estimate the cardinalities of $S_i(V')$ up to the i-th iterated neighbourhood for $i > 1$, provided $d^i = o(n)$ and thus $i = O(\log n/\log\log n)$.

Let $f = o(n/d)$. It follows from (2) and (1) (with $\delta = 4(\omega|Z|)^{-1/2}$) that, in the case $\omega \log n/2 \leq |Z| \leq f$, with probability at least $1 - n^{-3}$

$$|N_1(Z) \setminus Z| = d|Z| \left(1 + O\left(d|Z|/n\right) + O\left((\omega|Z|)^{-1/2}\right)\right), \tag{4}$$

where the bounds in $O()$ are uniform for every Z in the range (but may depend on the choice of $f = o(n/d)$). Note that (4) implies

$$|N_1(Z)| = (d+1)|Z| \left(1 + O\left(d|Z|/n\right) + O\left((\omega|Z|)^{-1/2}\right)\right). \tag{5}$$

Our goal is to show that a.a.s. $Z = N_j(V')$ satisfies (4) for every V' with $|V'| \leq 2$ and all $j \in \{0, 1 \ldots, i-1\}$. Since the case $j = 0$ is covered by (3), we fix V' and $j \in \{1, 2, \ldots, i-1\}$ and proceed to bound from above the probability that j is the smallest index that does not satisfy the aforementioned property. In order to obtain an upper bound on this probability, we condition on the event that (4) (and thus (5)) holds for $Z = N_k(V')$ and all $0 \leq k \leq j-1$. In particular, for all such k, we have

$$|N_{k+1}(V')| = (d+1)|N_k(V')| \left(1 + O\left(d^{k+1}/n\right) + O\left((\omega d^{-k/2})\right)\right).$$

Since $j < i = O(\log n/\log\log n)$ and $\sqrt{\omega} \leq (\log n)^2(\log\log n)$, the cumulative multiplicative error term in the estimation of $|N_j(V')|$ above is

$$\begin{aligned}(1+&O(d/n) + O(1/\sqrt{\omega})) \prod_{k=1}^{j-1} \left(1 + O\left(d^{k+1}/n\right) + O\left(\omega^{-1/2} d^{-k/2}\right)\right) \\ &= (1 + O(1/\sqrt{\omega}) + O(d^j/n)) \prod_{k=6}^{j-4} \left(1 + O\left(\log^{-3} n\right)\right) = (1 + O(1/\sqrt{\omega}) + O(d^j/n)).\end{aligned}$$

Therefore, deterministically from our assumptions,

$$|N_j(V')| = (d+1)^{j-1} d(1 + O(1/\sqrt{\omega}) + O(d^j/n)) = d^j(1 + O(1/\sqrt{\omega}) + O(d^i/n)), \tag{6}$$

which in particular meets the condition $|N_j(V')| = o(n/d)$ required to bound the probability that (4) fails for $Z = N_j(V')$. We conclude that this probability is at most n^{-3}. Taking a union bound over all $O(n^2 \log n)$ choices for V' and j, shows that a.a.s. $Z = N_j(V')$ satisfies (4) for every V' and j in the desired range. As a result, (6) is a.a.s. valid for all $1 \leq j \leq i$. This yields the first statement of the lemma, since

$$N_i(V') \setminus N_{i-1}(V') \subseteq S_i(V') \subseteq N_i(V'),$$

and the proof is complete. $\square$

4. Generalization—the case $k \geq 2$

4.1. **Upper bound.** Lemma 2.1 can be easily generalized to get the following (deterministic) upper bound which implies an upper bound for (sufficiently dense) random graphs.

Lemma 4.1. *For any graph G and any $k \in \mathbb{N}$, we have*

$$\chi(G^{2k}) \leq \Delta \sum_{i=0}^{2k-1} (\Delta - 1)^i + 1 \leq \Delta(\Delta-1)^{2k-1} \sum_{i=0}^{\infty} (\Delta-1)^{-i} + 1 = (\Delta-1)^{2k} \frac{\Delta}{\Delta - 2} + 1,$$

where $\Delta = \Delta(G)$ is the maximum degree of G.

Corollary 4.2. *Suppose that $d = pn \gg \log n$. Then, for any $k = k(n) \in \mathbb{N}$ a.a.s.*

$$\chi(\mathcal{G}(n,p)^{2k}) \leq (1+o(1))(pn)^{2k}.$$

Proof. Let $G \in \mathcal{G}(n,p)$. It follows from Chernoff's bound (1) that a.a.s. $\Delta = \Delta(G) = d(1 + O(\sqrt{\log n/d})) \sim d$, so we might want to use Lemma 4.1. However, this time we need to ensure that for sparse graphs the error term does not cumulate. As the diameter of G is a.a.s. $O(\log n/\log\log n)$, the error term is equal to

$$\left(1 + O\left(\sqrt{\frac{\log n}{d}}\right)\right)^{O(\log n/\log\log n)} = 1 + O\left(\sqrt{\frac{\log^3 n}{d \log^2 \log n}}\right) \sim 1,$$

provided that $d \gg (\log n)^3/(\log\log n)^2$. Fortunately, for sparser graphs, we can estimate the size of the $(2k)$-th neighbourhood directly from Lemma 3.1 to extend the result for the whole range of p. Indeed, if $2k$ is such that $d^{2k} = o(n)$, then we get immediately from Lemma 3.1 that a.a.s. for any v, $|N_{2k}(v)| \sim d^{2k}$. Hence, $\chi(G^{2k}) \leq (1+o(1))d^{2k}$, by a trivial greedy colouring argument. If $2k$ is such that $d^{2k-1} = o(n)$ but $d^{2k} = \Omega(n)$, then, arguing as before, we get that a.a.s. for any v, $|N_{2k-1}(v)| \sim d^{2k-1}$ and so, since a.a.s. $\Delta \sim d$, it follows that a.a.s. $|N_{2k}(v)| \leq (1+o(1))d^{2k}$ for any v. In the remaining cases, $d^{2k} \gg n$ and so the bound trivially (and deterministically) holds. $\square$

4.2. **Lower bound.** As before, we will deal with dense and sparse random graphs independently, starting with the dense case.

Lemma 4.3. *Let $k = k(n) \in \mathbb{N} \setminus \{1\}$ and $\varepsilon \in (0, 1/3)$. Suppose that $pn \gg \log n$ and that*

$$((2+\varepsilon)n\log n)^{1/(2k)} \leq pn = o(n^{1/(k-1)}).$$

Then, a.a.s.

$$\chi_i^k(\mathcal{G}(n,p)) = n.$$

Proof. The proof is a generalization of the argument used to prove Lemma 2.3. We are going to show that a.a.s. for any two vertices of $G = (V,E) \in \mathcal{G}(n,p)$ there exists a vertex that is at distance k from both of them. This will finish the proof as it implies that no two vertices can have the same colour assigned.

Let X be the random variable counting the number of pairs of vertices x, y for which there is no $z \in V \setminus N_{k-1}(\{x,y\})$ such that z has both a neighbour in $S_{k-1}(x) \setminus S_{k-1}(y)$

and a neighbour in $S_{k-1}(y) \setminus S_{k-1}(x)$. Suppose first that $d^k/n \le 1/\log n = o(1)$. For a given pair x, y, by Lemma 3.1, the probability that no suitable z can be found is at most

$$\begin{aligned}
&\Big(1 - \big(1-(1-p)^{|S_{k-1}(x)\setminus S_{k-1}(y)|}\big)\big(1-(1-p)^{|S_{k-1}(y)\setminus S_{k-1}(x)|}\big)\Big)^{n-|N_{k-1}(\{x,y\})|} \\
&\quad = \Big(1 - \big(1-\exp(-(1+o(1))pd^{k-1})\big)^2\Big)^{(1+o(1))n} \\
&\quad = \Big(1 - \big(1-\exp(-(1+o(1))d^k/n)\big)^2\Big)^{(1+o(1))n} \\
&\quad = \Big(1 - \big((1+o(1))d^k/n\big)^2\Big)^{(1+o(1))n} \\
&\quad \le \exp\Big(-(1+o(1))d^{2k}/n\Big).
\end{aligned}$$

(The last equality follows from the assumption that $d^k/n = o(1)$.) We get

$$\mathbb{E}X \le \binom{n}{2} \exp(-(1+o(1))d^{2k}/n) \le n^2 \exp\left(-(2+\varepsilon+o(1))\log n\right) = o(1).$$

Similarly, for $d^k/n > 1/\log n$, we get

$$\begin{aligned}
\mathbb{E}X &\le \binom{n}{2}\Big(1-((1+o(1))/\log n)^2\Big)^{(1+o(1))n} \\
&\le \exp(2\log n - (1+o(1))n/\log^2 n) = o(1).
\end{aligned}$$

In either case, the result holds by the first moment method. □

Now, we move to the sparse case.

Lemma 4.4. *Let $k = k(n) \in \mathbb{N} \setminus \{1\}$ and $\varepsilon \in (0, 1/3)$. Suppose that $\log n \ll pn < ((2+\varepsilon)n\log n)^{1/(2k)}$. Then, a.a.s.*

$$\chi_i^k(\mathcal{G}(n,p)) \ge \frac{(pn)^{2k}}{10k\log(pn)}.$$

Proof. The proof is a generalization of the argument used to prove Lemma 2.4. Let $G = (V, E) \in \mathcal{G}(n, p)$. We are going to show that a.a.s. for every set $S \subseteq V$ of cardinality $s = \lceil (10kn\log(pn))/(pn)^{2k} \rceil$ there exists a vertex $v = v(S) \in V \setminus N_{k-1}(S)$ that is at distance k from at least two vertices of S. This will imply that $\chi_i(G) \ge n/(s-1) \ge (pn)^{2k}/(10k\log(pn))$, and the proof will be finished.

Let S be any set of size s and let $v \in V \setminus N_{k-1}(S)$. We with to estimate $q = q(S, v)$, the probability that there is at most one vertex in S that is at distance k from v. For any $A \subseteq S$, let $E_{A,v}$ be the event that $S_k(v) \cap S = A$ (that is, A is the set of vertices of S that are at distance k from v). It follows from Bonferroni's inequality that

$$\begin{aligned}
q &= 1 - \sum_{A\subseteq S, |A|=2} \Pr(E_{A,v}) + \sum_{A\subseteq S, |A|=3} \Pr(E_{A,v}) - \sum_{A\subseteq S, |A|=4} \Pr(E_{A,v}) + \ldots \\
&\le 1 - \sum_{A\subseteq S, |A|=2} \Pr(E_{A,v}) + \sum_{A\subseteq S, |A|=3} \Pr(E_{A,v}).
\end{aligned}$$

Moreover, it is straightforward to show that

$$q \le 1-(1+o(1))\sum_{A\subseteq S,|A|=2} \Pr(E_{A,v}).$$

(See calculations below.) Using Lemma 3.1, let us observe that

$$\begin{aligned}
\sum_{A\subseteq S,|A|=2} \Pr(E_{A,v}) &\ge \sum_{x,y\in S}\left(1-(1-p)^{|N_{k-1}(x)\setminus N_{k-1}(y)|}\right)\left(1-(1-p)^{|N_{k-1}(y)\setminus N_{k-1}(x)|}\right)\\
&\quad \cdot(1-p)^{|N_{k-1}(S)|-|N_{k-1}(x)\setminus N_{k-1}(y)|-|N_{k-1}(y)\setminus N_{k-1}(x)|}\\
&= \binom{s}{2}\left(1-\exp(-(1+o(1))p(pn)^{k-1})\right)^2\exp(-O(ps(pn)^{k-1}))\\
&= (1+o(1))\binom{s}{2}\left(p(pn)^{k-1}\right)^2 = (1+o(1))\binom{s}{2}(pn)^{2k}/n^2,
\end{aligned}$$

since $ps(pn)^{k-1} = O(\log(pn)/(pn)^k) = o(1)$. Hence, we get the following estimation for $r = r(S)$, the probability that all vertices in $V \setminus N_{k-1}(S)$ have at most one vertex in S in their k-th neighbourhood:

$$\begin{aligned}
r &\le \prod_{v\in V\setminus N_{k-1}(S)}\left(1-(1+o(1))\sum_{A\subseteq S,|A|=2}\Pr(E_{A,v})\right)\\
&\le \left(1-(1+o(1))\binom{s}{2}(pn)^{2k}/n^2\right)^{n-O(s(pn)^{k-1})}\\
&= \exp\left(-(1+o(1))\binom{s}{2}(pn)^{2k}/n\right).
\end{aligned}$$

Finally, let X_s be the random variable counting the number of sets of vertices of cardinality s for which there is no vertex at distance k from at least two vertices in S. It follows that

$$\begin{aligned}
\mathbb{E}X_s = \sum_{S\subseteq V,|S|=s} r(S) &\le \binom{n}{s}\exp\left(-(1+o(1))\binom{s}{2}(pn)^{2k}/n\right)\\
&\le \left(\frac{en}{s}\right)^s\exp\left(-(1+o(1))\frac{s(s-1)}{2}(pn)^{2k}/n\right)\\
&\le (pn)^{2ks}\exp\left(-(1+o(1))\frac{s^2}{4}(pn)^{2k}/n\right).
\end{aligned}$$

The last inequality follows from the fact that $s \ge (10kn\log(pn))/(pn)^{2k} \ge 5/(2+1/3)+o(1) \ge 2$ (since $pn < ((2+1/3)n\log n)^{1/(2k)}$) and so $s(s-1)/2 \ge s^2/4$. We get that

$$\begin{aligned}
\mathbb{E}X_s &\le \exp\left(s\left(2k\log(pn)-(1+o(1))\frac{s}{4}(pn)^{2k}/n\right)\right)\\
&= \exp\left(-\left(\frac{1}{2}+o(1)\right)sk\log(pn)\right) = o(1).
\end{aligned}$$

The result holds by the first moment method. □

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