

# To Catch a Falling Robber

William B. Kinnersley\*, Paweł Prałat<sup>†</sup> and Douglas B. West<sup>‡</sup>

## Abstract

We consider a Cops-and-Robber game played on the subsets of an  $n$ -set. The robber starts at the full set; the cops start at the empty set. In each round, each cop moves up one level by gaining an element, and the robber moves down one level by discarding an element. The question is how many cops are needed to ensure catching the robber when the robber reaches the middle level. Alan Hill posed the problem and provided a lower bound of  $2^{n/2}$  for even  $n$  and  $\binom{n}{\lceil n/2 \rceil} 2^{-\lfloor n/2 \rfloor}$  (which is asymptotic to  $2^{\lceil n/2 \rceil} / \sqrt{\pi n/2}$ ) for odd  $n$ . Until now, no nontrivial upper bound was known. In this paper, we prove an upper bound that is within a factor of  $O(\ln n)$  of this lower bound.

Keywords: Cops-and-robber game; cop number; hypercube;  $n$ -dimensional cube

## 1 Introduction

The game of *Cops-and-Robber* is a pursuit game on a graph. In the classical form, there is one robber and some number of cops. The players begin by occupying vertices, first the cops and then the robber; multiple cops may simultaneously occupy the same vertex. In each subsequent round, each cop and then the robber can move along an edge to an adjacent vertex. The cops win if at some point there is a cop occupying the same vertex as the robber. The *cop number* of a graph  $G$ , written  $c(G)$ , is the least number of cops that can guarantee winning (all players always know each others' positions).

The game of Cops-and-Robber was independently introduced by Quilliot [15] and by Nowakowski and Winkler [12]; both papers characterized the graphs with cop number 1. The cop number as a graph invariant was then introduced by Aigner and Fromme [1]. Analysis of the cop number is the central problem in the study of the game and often is quite challenging. The foremost open problem in the field is Meyniel's conjecture that  $c(G) = O(\sqrt{n})$  for every  $n$ -vertex connected graph  $G$  (first published in [5]). This problem has a relatively long history.

---

\*Department of Mathematics, University of Rhode Island, Kingston, RI 02881, billk@uri.edu

<sup>†</sup>Department of Mathematics, Ryerson University, Toronto, ON, Canada M5B 2K3, pralat@ryerson.ca. Research supported by NSERC and Ryerson University

<sup>‡</sup>Departments of Mathematics, Zhejiang Normal University, Jinhua, China 321004, and University of Illinois, Urbana, IL 61801, west@math.uiuc.edu. Research supported by Recruitment Program of Foreign Experts, 1000 Talent Plan, State Administration of Foreign Experts Affairs, China.

At present we know only that the cop number is at most  $n2^{-(1+o(1))\sqrt{\log_2 n}}$  (still in  $n^{1-o(1)}$ ) for any connected graph on  $n$  vertices. This result was obtained independently by Lu and Peng [9], Scott and Sudakov [16], and Frieze, Krivelevich, and Loh [6] using probabilistic tools. For evidence supporting Meyniel’s conjecture, it is natural to check first whether random graphs provide easy counterexamples. It is known that Meyniel’s conjecture passes this test for binomial random graphs [4, 2, 10, 13] and for random  $d$ -regular graphs [14]: for connected graphs in these models, the conjecture holds asymptotically almost surely. For more background on Cops-and-Robber, see [3].

We consider a variant of the Cops-and-Robber game on a hypercube, introduced in the thesis of Alan Hill [7]. This variant restricts the initial positions and the allowed moves. The  $n$ -dimensional hypercube is the graph with vertex set  $\{0, 1\}^n$  (the set of binary  $n$ -tuples) in which vertices are adjacent if and only if they differ in one coordinate. View the vertices as subsets of  $\{1, \dots, n\}$ , and let the  $k$ th level consist of the  $k$ -sets – that is, the vertices whose size as subsets is  $k$ . We view  $\emptyset$  as the “bottom” of the hypercube and  $\{1, \dots, n\}$  as the “top”, and we say that  $S$  lies *below*  $T$  when  $S \subseteq T$ .

The robber starts at the full set  $\{1, \dots, n\}$ ; the cops start at the empty set  $\emptyset$ . On the  $k$ th round, the cops all move from level  $k - 1$  to level  $k$ , and then the robber moves from level  $n + 1 - k$  to level  $n - k$ . If the cops catch the robber, then they do so on round  $\lceil n/2 \rceil$  at level  $\lceil n/2 \rceil$ , when they move if  $n$  is odd, and by the robber moving onto them if  $n$  is even.

Let  $c_n$  denote the minimum number of cops that can guarantee winning the game. Hill [7] provided the lower bound  $2^{n/2}$  for even  $n$  and  $\binom{n}{\lceil n/2 \rceil} 2^{-\lfloor n/2 \rfloor}$  for odd  $n$ ; the former bound exceeds the latter by a factor of  $\Theta(\sqrt{n})$ . Note that here the cops have in some sense only one chance to catch the robber, on the middle level. When the cops can chase the robber by moving both up and down, the value is much smaller, with the cop number of the  $n$ -dimensional hypercube graph being  $\lceil (n + 1)/2 \rceil$  [11].

We begin with a proof of Hill’s lower bound, since its ideas motivate our arguments. (The proof below is essentially Hill’s original proof, albeit presented more concisely.) We then prove our result: an upper bound within a factor of  $O(\ln n)$  of this lower bound.

**Theorem 1.1** ([7]).  $c_n \geq \begin{cases} 2^m, & n = 2m; \\ \binom{2m+1}{m+1} 2^{-m}, & n = 2m + 1. \end{cases}$

*Proof.* After each move by the robber, some cops may no longer lie below the robber. Such cops are effectively eliminated from the game. We call them *evaded cops*; cops not yet evaded are *surviving cops*.

Consider the robber strategy that greedily evades as many cops as possible with each move. Deleting an element from the set at the robber’s current position evades all cops whose set contains that element. On the  $k$ th round, the surviving cops sit at sets of size  $k$ , and the robber has  $n - k + 1$  choices of an element to delete. Since each surviving cop can

be evaded in  $k$  ways, the fraction of the surviving cops that the robber can evade on this move is at least  $\frac{k}{n-k+1}$ .

After the first  $m$  rounds, where  $m = \lfloor n/2 \rfloor$ , the fraction of the cops that survive is at most  $\prod_{i=1}^m \left(1 - \frac{i}{n-i+1}\right)$ . When  $n = 2m$ , we compute

$$\prod_{i=1}^m \left(1 - \frac{i}{2m-i+1}\right) = \prod_{i=1}^m \frac{2m-2i+1}{2m-i+1} = \frac{(2m)!}{(2m)! \cdot 2^m} = 2^{-m}.$$

When  $n = 2m + 1$ , we compute

$$\prod_{i=1}^m \left(1 - \frac{i}{2m-i+2}\right) = \prod_{i=1}^m \frac{2m-2i+2}{2m-i+2} = \frac{2^m m! (m+1)!}{(2m+1)!} = 2^m / \binom{2m+1}{m+1}.$$

For the cops to catch the robber, at least one surviving cop must remain after  $m$  moves; this requires at least  $2^m$  total cops when  $n = 2m$  and at least  $\binom{2m+1}{m+1} 2^{-m}$  when  $n = 2m + 1$ .  $\square$

A similarly randomized strategy for the cops should produce a good upper bound. However, it is difficult to control the deviations from expected behavior over all the cops together. Our strategy will group the play of the game into phases that enable us to give essentially the same bound on undesirable deviations in each phase.

## 2 The Upper Bound

If there are enough cops to cover the entire middle level, then the robber cannot sneak through. The size of the middle level is asymptotic to  $2^n / \sqrt{\pi n/2}$ . This trivial upper bound is roughly the square of the lower bound in Theorem 1.1. When  $n$  is odd, a slight improvement follows by observing that one only needs to block each  $(n+1)/2$ -set by reaching some  $(n-1)/2$ -set under it. More substantial improvements use the fact that as the robber starts to move, the family of sets needing to be protected shrinks.

Our upper bound on  $c_n$  is  $O(\ln n)$  times the lower bound in Theorem 1.1. We use a randomized strategy for the cops; it may or may not succeed in capturing the robber. However, with sufficiently many cops, the strategy succeeds *asymptotically almost surely* (or *a.a.s.*), that is, with probability tending to 1 as  $n$  tends to infinity. Consequently, some deterministic strategy for the cops (in response to the moves by the robber) wins the game.

To analyze our cop strategy, we need a version of the well-known Chernoff Bound:

**Theorem 2.1** ([8]). *Let  $X$  be a random variable expressed as the sum  $\sum_{i=1}^n X_i$  of independent indicator random variables  $X_1, \dots, X_n$ , where  $X_i$  is a Bernoulli random variable with expectation  $p_i$  (the expectations need not be equal). For  $0 \leq \varepsilon \leq 1$ ,*

$$\mathbb{P}[X \leq (1 - \varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2 \mathbb{E}[X]}{2}\right).$$

We are now ready to prove our result.

**Theorem 2.2.**  $c_n = \begin{cases} O(2^m \ln n), & n = 2m; \\ O(2^{-m} \binom{2m+1}{m+1} \ln n), & n = 2m + 1. \end{cases}$

*Proof.* We consider the case  $n = 2m$  first, returning later to the case  $n = 2m + 1$ .

We will specify the number of cops later. All the cops begin at  $\emptyset$ . Let  $R$  be the current set occupied by the robber. On his  $k$ th turn, for  $1 \leq k \leq m$ , each surviving cop at set  $C$  chooses the next element for his set uniformly at random from among  $R - C$ . We claim that, regardless of how the robber moves, this cop strategy succeeds a.a.s.

To facilitate analysis of the cops' strategy, we introduce some notation and terminology. Consider an instance of the game. We say that this instance satisfies *property*  $P(t, a)$  if, after  $t$  rounds, every  $m$ -set below the robber also has at least  $a$  cops at or below it. Intuitively, the  $m$ -sets below the robber are the places where the robber can potentially be captured; property  $P(t, a)$  means that each of them can be reached by at least  $a$  cops.

To show that the cop strategy a.a.s. captures the robber, we will show that, no matter how the robber plays, a.a.s. property  $P(t_i, a_i)$  holds for specific choices of  $t_i$  and  $a_i$ . Let  $r = \lceil \log_2 \log_2 n \rceil$ , and for  $i \in \{0, \dots, r\}$  let  $s_i = 2^{r-i}$  and  $t_i = m - s_i$ . Furthermore, let

$$a_i = 1600 \left( \prod_{j=1}^i (1 - \varepsilon_j) \right) 2^{s_i} \ln n, \quad \text{where } \varepsilon_j = \sqrt{s_j/2^{s_j}}.$$

In particular,  $a_0 = 1600 \cdot 2^{2^r} \ln n$ . Note that always

$$\begin{aligned} \prod_{j=1}^i (1 - \varepsilon_j) &\geq \prod_{j=1}^r (1 - \varepsilon_j) \\ &\geq \exp \left( -2 \sum_{j=1}^r \varepsilon_j \right) \\ &\geq \exp \left( -2 \left( \sqrt{2^0/2^{2^0}} + \sqrt{2^1/2^{2^1}} + \sqrt{2^2/2^{2^2}} (1 + 1/2 + 1/4 + \dots) \right) \right) \\ &= \exp(-2\sqrt{2} - 2) > 1/200, \end{aligned}$$

and hence  $a_i \geq 8 \cdot 2^{s_i} \ln n$ . (Above, the second inequality uses the fact that  $1 - x \geq \exp(-2x)$  whenever  $0 \leq x \leq 1/\sqrt{2}$ , while the third inequality uses the observation that  $\varepsilon_{j-1} \leq \varepsilon_j/2$  for  $0 \leq j \leq r - 2$ .)

We play the game with  $\lceil 3200 \cdot 2^m \ln n \rceil$  cops. We claim that a.a.s. property  $P(t_i, a_i)$  holds for all  $i$  in  $\{0, \dots, r\}$ . We also claim that a.a.s. property  $P(m, 1)$  holds. This ensures that in the final round the cops can cover all vertices where the robber can move; hence they win.

We break the game into  $r + 2$  *phases*. Phase 0 consists of rounds 1 through  $t_0$ . For  $i \in \{1, \dots, r\}$ , Phase  $i$  consists of rounds  $t_{i-1} + 1$  through  $t_i$ . Phase  $r + 1$  consists of the single

round  $t_r + 1$ . Our analysis is inductive. For Phase 0, we show that a.a.s. property  $P(t_0, a_0)$  holds. When considering Phase  $i$  for  $1 \leq i \leq r$ , we assume that property  $P(t_{i-1}, a_{i-1})$  holds and show that a.a.s. property  $P(t_i, a_i)$  also holds. Finally, for Phase  $r + 1$ , we assume that property  $P(t_r, a_r)$  holds and show that a.a.s. the cops capture the robber.

We begin with Phase 0. We claim that property  $P(t_0, a_0)$  holds with probability at least  $1 - 1/n$ , no matter how the robber moves. Fix a sequence of moves for the robber in the first  $t_0$  rounds of the game, and fix a set  $S$  with  $|S| = m$  that remains below the robber. A particular cop remains below  $S$  if and only if his position contains only elements of  $S$ . In round  $i$ , each cop below  $S$  has already added  $i - 1$  such elements, and  $m - i + 1$  others remain. Since each surviving cop chooses a new element uniformly from  $2m - 2i + 2$  possibilities, the probability that a cop below  $S$  remains below  $S$  is  $\frac{m-i+1}{2m-2i+2}$ , which equals  $1/2$ . Thus, a given cop remains below  $S$  after the first  $t_0$  rounds with probability  $2^{-t_0}$ .

Consequently, the number of cops remaining below  $S$  after  $t_0$  rounds is a random variable  $X$  with the binomial distribution  $\text{Bin}(\lceil 3200 \cdot 2^m \ln n \rceil, 2^{-t_0})$ . Recalling that  $t_0 = m - s_0$  and that  $s_0 = 2^r \geq \log_2 n$ , we have

$$\mathbb{E}[X] \geq 3200 \cdot 2^m \ln n \cdot 2^{-t_0} = 3200 \cdot 2^{s_0} \ln n = 3200 \cdot 2^{2^r} \ln n = 2a_0.$$

The Chernoff Bound now yields

$$\mathbb{P}(X \leq a_0) \leq \mathbb{P}\left(X \leq \frac{\mathbb{E}[X]}{2}\right) \leq \exp\left(-\frac{(1/2)^2 \mathbb{E}[X]}{2}\right) < \exp(-3n \ln n).$$

Thus, the probability that fewer than  $a_0$  cops remain below  $S$  is less than  $\exp(-3n \ln n)$ . The number of such sets  $S$  below the robber is less than  $2^n$ , which is less than  $\exp(n \ln n)$ . By the Union Bound, the probability that some  $m$ -set below the robber has fewer than  $a_0$  cops below it is thus less than  $\exp(-2n \ln n)$ . That is, for one sequence of moves by the robber, property  $P(t_0, a_0)$  fails to hold with probability at most  $\exp(-2n \ln n)$ . The number of possible move sequences by the robber in Phase 0 is less than  $n^{t_0}$ , which in turn is less than  $\exp(n \ln n)$ . Again using the Union Bound, the probability that some robber strategy causes property  $P(t_0, a_0)$  to fail is less than  $\exp(-n \ln n)$ . Thus property  $P(t_0, a_0)$  holds with probability more than  $1 - \exp(-n \ln n)$ , which is more than  $1 - 1/n$ .

Next consider Phase  $i$  with  $1 \leq i \leq r$ , consisting of rounds  $t_{i-1} + 1$  through  $t_i$ . Under the assumption that property  $P(t_{i-1}, a_{i-1})$  holds, we claim that property  $P(t_i, a_i)$  also holds with probability at least  $1 - 1/n$ . The argument is similar to that for Phase 0. Fix a sequence of moves for the robber in rounds  $t_{i-1} + 1$  through  $t_i$ , and fix an  $m$ -set  $S$  that remains below the robber after round  $t_i$ . Again a cop below  $S$  on a given round remains below  $S$  after that round with probability  $1/2$ .

By assumption, at least  $a_{i-1}$  cops sat below  $S$  at the beginning of Phase  $i$ ; the number of cops remaining below  $S$  at the end of Phase  $i$  is thus bounded below by the random variable

$X$  with binomial distribution  $\text{Bin}(\lceil a_{i-1} \rceil, 2^{-(t_i - t_{i-1})})$ . Hence

$$\mathbb{E}[X] \geq a_{i-1} 2^{t_{i-1} - t_i} = a_{i-1} 2^{s_i - s_{i-1}} = a_i \cdot \frac{a_{i-1} \cdot 2^{-s_{i-1}}}{a_i \cdot 2^{-s_i}} = \frac{a_i}{1 - \varepsilon_i}.$$

This time, the Chernoff Bound yields

$$\begin{aligned} \mathbb{P}(X \leq a_i) &\leq \mathbb{P}(X \leq (1 - \varepsilon_i)\mathbb{E}[X]) \leq \exp\left(-\frac{\varepsilon_i^2 \cdot \mathbb{E}[X]}{2}\right) \leq \exp\left(-\frac{\varepsilon_i^2 \cdot a_i}{2}\right) \\ &\leq \exp\left(-\frac{\varepsilon_i^2 \cdot 8 \cdot 2^{s_i} \ln n}{2}\right) = \exp(-4s_i \ln n). \end{aligned}$$

At the start of Phase  $i$ , the robber occupies level  $n - t_{i-1}$ . At this time, the number of  $m$ -sets that lie below the robber is  $\binom{n - t_{i-1}}{m}$ . This simplifies to  $\binom{m + s_{i-1}}{m}$ , which is at most  $n^{s_{i-1}}$ ; since  $s_{i-1} = 2s_i$ , this is at most  $\exp(2s_i \ln n)$ . Likewise, the number of move sequences available to the robber during Phase  $i$  is at most  $n^{s_{i-1} - s_i}$ , which simplifies to  $\exp(s_i \ln n)$ . Applying the Union Bound twice, as in Phase 0, we see that property  $P(t_i, a_i)$  fails with probability at most  $\exp(-s_i \ln n)$ . Hence  $P(t_i, a_i)$  holds with probability at least  $1 - \exp(-s_i \ln n)$ , which is at least  $1 - 1/n$ .

Finally, we show that if  $P(t_r, a_r)$  holds, then  $P(m, 1)$  holds with probability at least  $1 - 1/n$ . Recall that  $t_r = m - 1$  and that  $a_r \geq 16 \ln n$ . Each cop chooses from two possible moves, each leading to an  $m$ -set. The number of cops that remain below an  $m$ -set  $S$  is bounded from below by the random variable  $X$  with distribution  $\text{Bin}(\lceil a_r \rceil, 1/2)$ . Now

$$\mathbb{P}(X = 0) = 2^{-\lceil a_r \rceil} \leq 2^{-16 \ln n} \leq \frac{1}{n^2},$$

so the probability that no cop reaches  $S$  is at most  $1/n^2$ . There are  $m + 1$  choices for  $S$ ; by the Union Bound,  $P(m, 1)$  fails with probability less than  $1/n$ . Hence  $P(m, 1)$  holds with probability at least  $1 - 1/n$ , as claimed.

To complete the proof, we now consider the full game. We want to show that a.a.s.  $P(m, 1)$  holds. The probability that  $P(m, 1)$  holds is bounded below by the probability that  $P(t_0, a_0), \dots, P(t_r, a_r)$ , and  $P(m, 1)$  all hold. We have shown that  $P(t_0, a_0)$  fails with probability at most  $1/n$ , that  $P(t_i, a_i)$  for  $1 \leq i \leq r$  fails with probability at most  $1/n$  when  $P(t_{i-1}, a_{i-1})$  holds, and that  $P(m, 1)$  fails with probability at most  $1/n$  when  $P(t_r, a_r)$  holds. By the Union Bound, the probability that some property in this list fails is bounded above by  $(r + 2)/n$ , which is at most  $2 \log_2 \log_2 n/n$  when  $n$  is sufficiently large. Thus the conjunction of these properties (and in particular, property  $P(m, 1)$ ) holds with probability at least  $1 - 2 \log_2 \log_2 n/n$ . This completes the proof for the case  $n = 2m$ .

When  $n = 2m + 1$ , we define property  $P(t, a)$  to mean that after  $t$  rounds, at least  $a$  cops sit below each  $(m + 1)$ -set that is below the robber. It now suffices to prove that  $P(m, 1)$  holds a.a.s., since any cop that remains below the robber at the beginning of round  $m + 1$  can capture him. The details of the argument are nearly identical to the previous case, and we omit them.  $\square$

We remark that the cops can play more efficiently by using an appropriate deterministic strategy in round  $m$ . This does not improve the asymptotics of our bound, but it does improve the leading constant.

## References

- [1] M. Aigner and M. Fromme, A game of cops and robbers, *Discrete Applied Mathematics* 8 (1984), 1–12.
- [2] B. Bollobás, G. Kun, I. Leader, Cops and robbers in a random graph, *Journal of Combinatorial Theory Series B* 103 (2013), 226–236.
- [3] A. Bonato and R.J. Nowakowski, *The Game of Cops and Robbers on Graphs*, American Mathematical Society, Providence, Rhode Island, 2011.
- [4] A. Bonato, P. Prałat, and C. Wang, Network Security in Models of Complex Networks, *Internet Mathematics* 4 (2009), 419–436.
- [5] P. Frankl, Cops and robbers in graphs with large girth and Cayley graphs, *Discrete Applied Mathematics* 17 (1987), 301–305.
- [6] A. Frieze, M. Krivelevich, and P. Loh, Variations on Cops and Robbers, *Journal of Graph Theory* 69 (2012), 383–402.
- [7] A. Hill, Cops and Robbers: Theme and Variations, PhD Thesis. *ProQuest Dissertations and Theses*, 2008.
- [8] S. Janson, T. Łuczak, A. Ruciński, *Random Graphs*, Wiley, New York, 2000.
- [9] L. Lu and X. Peng, On Meyniel’s conjecture of the cop number, *Journal of Graph Theory* 71 (2012), 192–205.
- [10] T. Łuczak and P. Prałat, Chasing robbers on random graphs: zigzag theorem, *Random Structures and Algorithms* 37 (2010), 516–524.
- [11] M. Maamoun and H. Meyniel, On a game of policemen and robber, *Discrete Applied Mathematics* 17 (1987), 307–309.
- [12] R.J. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Mathematics* 43 (1983) 235–239.
- [13] P. Prałat and N.C. Wormald, Meyniel’s conjecture holds for random graphs, *Random Structures and Algorithms*, 48(2) (2016), 396–421.
- [14] P. Prałat and N.C. Wormald, Meyniel’s conjecture holds for random  $d$ -regular graphs, manuscript.
- [15] A. Quilliot, Jeux et points fixes sur les graphes, Thèse de 3ème cycle, Université de Paris VI, 1978, 131–145.

- [16] A. Scott and B. Sudakov, A bound for the cops and robbers problem, *SIAM J. of Discrete Math* 25 (2011), 1438–1442.