ACQUAINTANCE TIME OF RANDOM GRAPHS NEAR CONNECTIVITY THRESHOLD

ANDRZEJ DUDEK AND PAWEŁ PRAŁAT

Abstract. Benjamini, Shinkar, and Tsur stated the following conjecture on the acquaintance time: asymptotically almost surely $\mathcal{AC}(G) \leq p^{-1} \log^{O(1)} n$ for a random graph $G \in G(n,p)$, provided that G is connected. Recently, Kinnersley, Mitsche, and the second author made a major step towards this conjecture by showing that asymptotically almost surely $\mathcal{AC}(G) = O(\log n/p)$, provided that G has a Hamiltonian cycle. In this paper, we finish the task by showing that the conjecture holds in the strongest possible sense, that is, it holds right at the time the random graph process creates a connected graph. Moreover, we generalize and investigate the problem for random hypergraphs.

1. Introduction

In this paper, we study the following graph process, which was recently introduced by Benjamini, Shinkar, and Tsur [5]. Let $G = (V, E)$ be a finite connected graph. We start the process by placing one *agent* on each vertex of G. Every pair of agents sharing an edge is declared to be *acquainted*, and remains so throughout the process. In each round of the process, we choose some matching M in G. (M need not be maximal; perhaps it is a single edge.) For each edge of M, we swap the agents occupying its endpoints, which may cause more agents to become acquainted. The *acquaintance time* of G, denoted by $\mathcal{AC}(G)$, is the minimum number of rounds required for all agents to become acquainted with one another.

It is clear that

$$\mathcal{AC}(G) \geq \frac{\binom{|V|}{2}}{|E|} - 1, \tag{1}$$

since $|E|$ pairs are acquainted initially, and at most $|E|$ new pairs become acquainted in each round. In [5], it was shown that always $\mathcal{AC}(G) = O(\frac{n^2}{\log n/\log\log n})$, where $n = |V|$. A small progress was made in [14] where it was proved that $\mathcal{AC}(G) = O(n^2/\log n)$. This general upper bound was recently improved in [3] to $\mathcal{AC}(G) = O(n^{3/2})$, which was conjectured in [5] and is tight up to a multiplicative constant. Indeed, for all functions $f : \mathbb{N} \to \mathbb{N}$ with $1 \leq f(n) \leq n^{3/2}$, there are families $\{G_n\}$ of graphs with $|V(G_n)| = n$ for all n such that $\mathcal{AC}(G_n) = \Theta(f_n)$. The problem is similar in flavour to the problems

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of Routing Permutations on Graphs via Matchings [2], Gossiping and Broadcasting [11], and Target Set Selection [13, 7, 17].

A *hypergraph* H is an ordered pair $H = (V, E)$, where V is a finite set (the *vertex set*) and E is a family of disjoint subsets of V (the *edge set*). A hypergraph $H = (V, E)$ is r*-uniform* if all edges of H are of size r. In this paper, we consider the acquaintance time of random r-uniform hypergraphs, including binomial random graphs. For a given $r \in \mathbb{N} \setminus \{1\}$, the *random r-uniform hypergraph* $H_r(n, p)$ has n labelled vertices from a vertex set $V = [n] := \{1, 2, \ldots, n\}$, in which every subset $e \subseteq V$ of size $|e| = r$ is chosen to be an edge of H randomly and independently with probability p. For $r = 2$, this model reduces to the well known and thoroughly studied model $G(n, p)$ of *binomial random graphs*. (See, for example, [6, 12], for more.) Note that $p = p(n)$ may tend to zero (and usually does) as n tends to infinity. All asymptotics throughout are as $n \to \infty$ (we emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded). We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity.

Let $G \in G(n, p)$ with $p = p(n) \geq (1 - \varepsilon) \log n/n$ for some $\varepsilon > 0$. Recall that $\mathcal{AC}(G)$ is defined only for connected graphs, and $\log n/n$ is the sharp threshold for connectivity in $G(n, p)$—see, for example, [6, 12] for more. Hence, we will not be interested in sparser graphs. It follows immediately from Chernoff's bound that a.a.s. $|E(G)| = (1 + o(1))\binom{n}{2}p$. Hence, from the trivial lower bound (1) we have that a.a.s. $\mathcal{AC}(G) = \Omega(1/p)$. Despite the fact that no non-trivial upper bound on $\mathcal{AC}(G)$ was known, it was conjectured in [5] that a.a.s. $\mathcal{AC}(G) = O(\operatorname{poly}\log(n)/p)$. Recently, Kinnersley, Mitsche, and the second author of this paper made a major step towards this conjecture by showing that a.a.s. $\mathcal{AC}(G) = O(\log n/p)$, provided that G has a Hamiltonian cycle, that is, when $pn - \log n - \log\log n \to \infty$. (See [14] for more.)

In this paper, we finish the task by showing that the conjecture holds above the threshold for connectivity.

Theorem 1.1. *Suppose that $p = p(n)$ is such that $pn - \log n \to \infty$. For $G \in G(n, p)$, a.a.s.*

$$\mathcal{AC}(G) = O\left(\frac{\log n}{p}\right).$$

In fact, we prove the conjecture in the strongest possible sense, that is, we show that it holds right at the time the random graph process creates a connected graph. Before we state the result, let us introduce one more definition. We consider the Erdős-Rényi *random graph process*, which is a stochastic process that starts with n vertices and no edges, and at each step adds one new edge chosen uniformly at random from the set of missing edges. Formally, let $N = \binom{n}{2}$ and let $e_1, e_2, \ldots, e_N$ be a random permutation of the edges of the complete graph K_n. The graph process consists of the sequence of random graphs $(\mathcal{G}(n, m))_{m=0}^{N}$, where $\mathcal{G}(n, m) = (V, E_m)$, $V = [n]$, and $E_m = \{e_1, e_2, \ldots, e_m\}$. It is clear that $\mathcal{G}(n, m)$ is a graph taken uniformly at random from the set of all graphs on n vertices and m edges. (As before, see, for example, [6, 12] for more details.)

Note that in the missing window in which the conjecture is not proved, we have $p = (1 + o(1)) \log n/n$ and so an upper bound of $O(\log n/p)$ is equivalent to $O(n)$. Hence, it is enough to show the following result, which implies Theorem 1.1 as discussed at the beginning of Section 3.

Theorem 1.2. *Let M be a random variable defined as follows:*

$$M = \min\{m : \mathcal{G}(n, m) \text{ is connected}\}.$$

Then, for $G \in \mathcal{G}(n, M)$, a.a.s.

$$\mathcal{AC}(G) = O(n).$$

Fix $r \in \mathbb{N}\setminus\{1\}$. For an r-uniform hypergraph $H = (V, E)$, there are a number of ways one can generalize the problem. At each step of the process, we continue choosing some matching M of agents and swapping the agents occupying its endpoints; two agents can be matched if there exists $e \in E$ such that both agents occupy vertices of e. However, this time we have more flexibility in the definition of being acquainted. We can fix $k \in \mathbb{N}$ such that $2 \le k \le r$, and every k-tuple of agents sharing an edge is declared to be *k-acquainted*, and remains so throughout the process. The *k-acquaintance time* of H, denoted by $\mathcal{AC}_r^k(H)$, is the minimum number of rounds required for all k-tuples of agents to become acquainted. Clearly, for any graph G, $\mathcal{AC}_2^2(G) = \mathcal{AC}(G)$.

We show the following result for random r-uniform hypergraphs. Note that $p = \frac{(r-1)!\log n}{n^{r-1}}$ is the sharp threshold for connectivity so the assumption for the function p in the statement of the result is rather mild but could be weakened.

Theorem 1.3. *Let $r \in \mathbb{N} \setminus \{1\}$, let $k \in \mathbb{N}$ be such that $2 \le k \le r$, and let $\varepsilon > 0$ be any real number. Suppose that $p = p(n)$ is such that $p \ge (1+\varepsilon)\frac{(r-1)!\log n}{n^{r-1}}$. For $H \in H_r(n, p)$, a.a.s.*

$$\mathcal{AC}_r^k(H) = \Omega\left(\max\left\{\frac{1}{pn^{r-k}}, 1\right\}\right) \quad \text{and} \quad \mathcal{AC}_r^k(H) = O\left(\max\left\{\frac{\log n}{pn^{r-k}}, 1\right\}\right).$$

Throughout the paper, we will be using the following concentration inequalities. Let $X \in \mathrm{Bin}(n, p)$ be a random variable with the binomial distribution with parameters n and p. Then, a consequence of Chernoff's bound (see e.g. [12, Corollary 2.3]) is that

$$\mathbb{P}(|X - \mathbb{E}X| \ge \varepsilon\mathbb{E}X) \le 2\exp\left(-\frac{\varepsilon^2\mathbb{E}X}{3}\right) \tag{2}$$

for $0 < \varepsilon < 3/2$. This inequality will be used many times but at some point we will also apply the bound of Bernstein (see e.g. [12, Theorem 2.1]) that for every $x > 0$,

$$\mathbb{P}(X \ge (1 + x)\mathbb{E}X) \le \exp\left(-\frac{x^2\mathbb{E}X}{2(1 + x/3)}\right). \tag{3}$$

The paper is structured as follows. In Section 2, we investigate the existence of long paths in $H_r(n, p)$. Our result works for any $r \ge 2$ and will be used in both graphs and hypergraphs. In Section 3, we deal with random graphs ($r = 2$) and random hypergraphs ($r \ge 3$) are dealt in Section 4. We finish the paper with a few remarks.

2. Existence of long paths in $H_r(n,p)$

For positive integers k, r and ℓ satisfying $\ell = k(r-1)+1$, we say that an r-uniform hypergraph L is a *loose path* of length ℓ if there exist an ordering of the vertex set, $(v_1, v_2, \ldots, v_\ell)$, and an ordering of the edge set, $(e_1, e_2, \ldots, e_k)$, such that each edge consists of r consecutive vertices, that is, for every i, $e_i = \{v_{(i-1)(r-1)+q} : 1 \le q \le r\}$, and each pair of consecutive edges intersects on a single vertex, that is, for every i, $e_i \cap e_{i+1} = \{v_{i(r-1)+1}\}$. Note that the ordering $(v_1, v_2, \ldots, v_\ell)$ of vertices can be used to completely describe L.

In this section we show that even very sparse random hypergraphs contain long loose paths. This generalizes a well-known result of Ajtai, Komlós and Szemerédi [1] that holds for random graphs.

Lemma 2.1. *For fixed $r \in \mathbb{N} \setminus \{1\}$ and real number $0 < \delta < 1$, there is a positive constant $c = c(r,\delta)$ such that for $H \in H_r(n,p)$ with $p = c/n^{r-1}$*

$$\mathbb{P}\left(H \text{ has a loose path of length at least } \delta n\right) \ge 1 - \exp(-n).$$

(For expressions such as δn that clearly have to be an integer, we round up or down but do not specify which: the reader can choose either one, without affecting the argument.)

Proof. We will apply a depth first search algorithm in order to explore vertices of $H = (V,E) \in H_r(n,p)$ extending some ideas from [15]. Set

$$c = c(r,\delta) = \frac{2(r-1)!\log 4}{\frac{1-\delta}{2(r-1)}\left(\frac{1-\delta}{2}\right)^{r-1}}.$$

(Let us note that we make no effort to optimize c here.) In every step of the process, V is partitioned into the following four sets:

- P – ordered set of vertices representing a loose path $(v_1, v_2, \ldots, v_\ell)$ of length ℓ,
- U – set of unexplored vertices that have not been reached by the search yet,
- W and $\tilde{W}$ – sets of explored vertices that no longer play a role in the process.

We start the process by assigning to P an arbitrary vertex v from V and setting $U = V \setminus \{v\}$, $W = \tilde{W} = \emptyset$. (Note that it is vacuously true that $L = (\{v\}, \emptyset)$, a hypergraph consisting of one vertex and no edge, is a loose path.) Suppose that at some point of the process we found a path $P = (v_1, v_2, \ldots, v_\ell)$ of length ℓ. If there exists a set $f = \{u_1, u_2, \ldots, u_{r-1}\} \subseteq U$ of size $r-1$ such that $f \cup \{v_\ell\} \in E$, then we extend the path by adding the edge $\{v_\ell, u_1, u_2, \ldots, u_{r-1}\}$ to the path; vertices of f are removed from U. (After this operation, $P = (v_1, v_2, \ldots, v_\ell, u_1, u_2, \ldots, u_{r-1})$ has length $\ell + (r-1)$.) Otherwise, that is, if no such set f can be found in U, then v_ℓ is moved from P to W and, depending on the value of ℓ, we have two cases. If $\ell \ne 1$ (which implies that, in fact, $\ell \ge r$), then vertices $v_{\ell-1}, v_{\ell-2}, \ldots, v_{\ell-(r-2)}$ are moved from P to $\tilde{W}$. (After this operation, $P = (v_1, v_2, \ldots, v_{\ell-(r-1)})$ has length $\ell-(r-1)$.) If $\ell = 1$, then we simply take an arbitrary vertex from U and restart the process by taking $P = (v)$ of length 1.

Now, let us assume that H has no loose path of length at least δn. Therefore, in every step of the algorithm $\ell < \delta n$. Moreover, the process ends with $U = \emptyset$ and

$|W|+|\tilde{W}| = |V|-|P| > (1-\delta)n$. Hence, since $(r-2)|W| \geq |\tilde{W}|$, we have $|W| > \frac{1-\delta}{r-1}n$. Observe that in each step of the process either the size of W increases by one or the size of U decreases by at most $r-1$ (actually, the size of U decreases either by 1 or $r-1$). Thus, at some stage we must have $|W| = \frac{1-\delta}{2(r-1)}n$ and

$$|U| = |V| - |P| - |W| - |\tilde{W}| > n - \delta n - |W| - (r-2)|W| = \frac{1-\delta}{2}n.$$

Furthermore, note that there is no edge e between U and W such that $|U \cap e| = r-1$ and $|W \cap e| = 1$.

For fixed sets U and W satisfying $|U| = \frac{1-\delta}{2}n$ and $|W| = \frac{1-\delta}{2(r-1)}n$, let $e(U,W)$ be the number of edges e between U and W in H for which $|U \cap e| = r-1$ and $|W \cap e| = 1$. Then,

$$\begin{aligned}\mathbb{P}(e(U,W)=0) &= (1-p)^{|W|\binom{|U|}{r-1}} \\ &\leq \exp\left(-(1+o(1))\frac{c}{n^{r-1}} \cdot \frac{1-\delta}{2(r-1)}n \cdot \left(\frac{1-\delta}{2}n\right)^{r-1} / (r-1)!\right) \\ &= \exp(-(1+o(1))2n\log 4),\end{aligned}$$

and consequently by taking the union over all disjoint subsets $U, W \subseteq V$ satisfying $|U| = \frac{1-\delta}{2}n$ and $|W| = \frac{1-\delta}{2(r-1)}n$, we obtain

$$\begin{aligned}\mathbb{P}\left(H \text{ has no loose path of length at least } \delta n\right) &\leq \mathbb{P}\left(\bigcup_{U,W} e(U,W) = 0\right) \\ &\leq 2^n 2^n \exp(-(1+o(1))2n\log 4) = \exp(-(1+o(1))n\log 4) \leq \exp(-n),\end{aligned}$$

as required. □

In order to prove Theorem 1.3 we will also need the following result about Hamiltonian paths in $H_r(n,p)$, which follows from a series of papers [10, 8, 9] devoted to loose Hamiltonicity of random hypergraphs. Observe that if an r-uniform hypergraph of order n contains a loose Hamiltonian path (that is, a loose path that covers each vertex of H), then necessarily $r-1$ must divide $n-1$.

Lemma 2.2 ([10, 8, 9])**.** *For fixed integer $r \geq 3$, let $p = p(n)$ be such that $pn^{r-1}/\log n$ tends to infinity together with n and $H \in H_r(n,p)$. Then, a.a.s. H has a loose Hamiltonian path provided that $r-1$ divides $n-1$.*

3. Random graphs

Let us start with the following two comments. First of all, note the following: whenever G_2 is a subgraph of G_1 on the same vertex set, $\mathcal{AC}(G_1) \leq \mathcal{AC}(G_2)$, since the agents in G_1 have more edges to use. Hence, Theorem 1.2 implies that for any $m \geq M$ we also have that a.a.s. $\mathcal{AC}(G) = O(n)$, $G \in \mathcal{G}(n,m)$.

Second of all, it is known that the two models ($G(n,p)$ and $\mathcal{G}(n,m)$) are in many cases asymptotically equivalent, provided $\binom{n}{2}p$ is close to m. For example, Proposition 1.12 in [12] gives us the way to translate results from $\mathcal{G}(n,m)$ to $G(n,p)$.

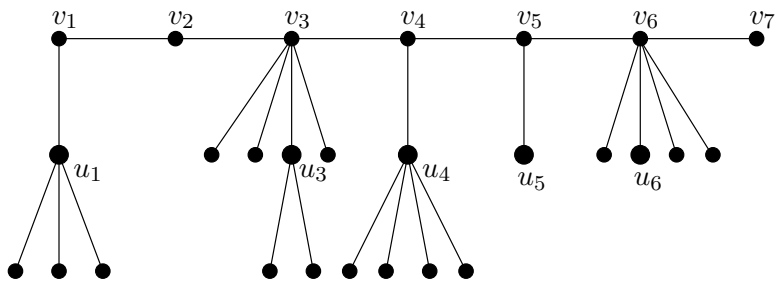

FIGURE 1. A good tree.

Lemma 3.1. *Let P be an arbitrary property, $p = p(n)$, and $c \in [0,1]$. If for every sequence $m = m(n)$ such that*

$$m = \binom{n}{2}p + O\left(\sqrt{\binom{n}{2}p(1-p)}\right)$$

it holds that $\mathbb{P}(\mathcal{G}(n,m) \in P) \to c$ as $n \to \infty$, then also $\mathbb{P}(G(n,p) \in P) \to c$ as $n \to \infty$.

Using this lemma, Theorem 1.2 implies immediately Theorem 1.1. Indeed, suppose that $p = p(n) = (\log n + \omega)/n$, where $\omega = \omega(n)$ tends to infinity together with n. One needs to investigate $\mathcal{G}(n,m)$ for $m = \frac{n}{2}(\log n + \omega + o(1))$. It is known that in this range of m, a.a.s. $\mathcal{G}(n,m)$ is connected (that is, a.a.s. $M < m$). Theorem 1.2 together with the first observation imply that a.a.s. $\mathcal{AC}(\mathcal{G}(n,m)) = O(n)$, and Theorem 1.1 follows from Lemma 3.1.

In order to prove Theorem 1.2, we will show that at the time when $\mathcal{G}(n,m)$ becomes connected (that is, at time T), a.a.s. it contains a certain spanning tree T with $\mathcal{AC}(T) = O(n)$. For that, we need to introduce the following useful family of trees. A tree T is *good* if it consists of a path $P = (v_1, v_2, \ldots, v_k)$ (called the *spine*), some vertices (called *heavy*) that form a set $\{u_i : i \in I \subseteq [k]\}$ that are connected to the spine by a perfect matching (that is, for every $i \in I$, u_i is adjacent to v_i). All other vertices (called *light*) are adjacent either to v_i for some $i \in [k]$ or to u_i for some $i \in I$ (see Figure 1 for an example).

We will use the following result from [5] (Claim 2.1 in that paper).

Claim 3.2 ([5])**.** *Let $G = (V,E)$ be a tree. Let $S, T \subseteq V$ be two subsets of the vertices of equal size $k = |S| = |T|$, and let $\ell = \max_{v \in S, u \in T} \mathrm{dist}(v,u)$ be the maximal distance between a vertex in S and a vertex in T. Then, there is a strategy of $\ell + 2(k-1)$ matchings that routes all agents from S to T.*

Now, we are ready to show that this family of trees is called good for a reason.

Lemma 3.3. *Let T be a good tree on n vertices. Then, $\mathcal{AC}(T) = O(n)$. Moreover, there exists an acquaintance strategy with $O(n)$ steps in which every agent visits every vertex of the spine.*

Proof. Let us call agents occupying the spine $P = (v_1, v_2, \ldots, v_k)$ *active*. First, we will show that there exists a strategy ensuring that within $6k$ rounds every active agent gets acquainted with everyone else and, moreover, that she visits every vertex of the spine.

For $1 \le i \le k-1$, let $e_i = v_i v_{i+1}$. The strategy is divided into $2k$ phases, each consisting of 3 rounds. On odd-numbered phases, swap active agents on all odd-indexed edges; then, swap every active agent adjacent to a heavy vertex (that is, agent occupying vertex v_i for some $i \in I$) with non-active agent (occupying u_i), and then repeat it again so that all active agents are back onto the spine. On even-numbered phases, swap agents on all even-indexed edges, followed by two rounds of swapping with non-active agents, as it was done before. This has the following effect. Agents that begin on odd-indexed vertices move "forward" in the vertex ordering, pause for one round at v_k, move "backward", pause again at v_1, and repeat; agents that begin on even-indexed vertices move backward, pause at v_1, move forward, pause at v_k, and repeat. Since T is good, every non-active agent was initially adjacent to some vertex of the spine or some heavy vertex. Hence, after $2k$ phases, each active agent has traversed the entire spine taking detours to heavy vertices, if needed; in doing so, she has necessarily passed by every other agent (regardless whether active or non-active).

It remains to show that agents can be rotated so that each agent is active at some point. Note that the diameter of T is $O(k)$. Partition the n agents into $\lceil n/k \rceil$ teams, each of size at most k. We iteratively route a team onto the spine, call them active, apply the strategy described earlier for the active team, and repeat until all teams have traversed the spine. Thus, by Claim 3.2, each iteration can be completed in $O(k)$ rounds, so the total number of rounds needed is $\lceil n/k \rceil \cdot O(k) = O(n)$. □

As we already mentioned, our goal is to show that at the time when $\mathcal{G}(n,m)$ becomes connected, a.a.s. it contains a good spanning tree T. However, it is easier to work with $G(n,p)$ model instead of $G(n,m)$. Lemma 3.1 provides us with a tool to translate results from $\mathcal{G}(n,m)$ to $G(n,p)$. The following lemma works the other way round, see, for example, (1.6) in [12].

Lemma 3.4. *Let P be an arbitrary property, let $m = m(n)$ be any function such that $m \le n \log n$, and take $p = p(n) = m/\binom{n}{2}$. Then,*

$$\mathbb{P}(\mathcal{G}(n,m) \in P) \le 3\sqrt{n \log n} \cdot \mathbb{P}(G(n,p) \in P).$$

We will also need the following lemma.

Lemma 3.5. *Consider $G = (V,E) \in G(n, \log\log\log n/n)$. Then, for every set A of size $0.99n$ the following holds. For a set B of size $O(n^{0.03})$ taken uniformly at random from $V \setminus A$, B induces a graph with no edge with probability $1 - o((n \log n)^{-1/2})$.*

Since the statement of the lemma is slightly technical, let us explain it a bit more. There are $\binom{n}{0.99n}$ sets of size $0.99n$ and, since *all* of them have the desired property, let us focus on any set A of such cardinality. Clearly, one should expect many sets of size $O(n^{0.03})$ lying outside of A that induce some edges. However, we claim that a random set $B \subseteq V \setminus A$ of such cardinality induces no edge with the desired probability.

Proof. Fix a vertex $v \in V$. The expected degree of v is $(1+o(1)) \log\log\log n$. It follows from Bernstein's bound (3), applied with $x = c \log n / \log\log\log n$ for c large enough,

that with probability $o(n^{-2})$, v has degree larger than, say, $2c\log n$. Thus, the union bound over all vertices of G implies that with probability $1-o((n\log n)^{-1/2})$ all vertices have degrees at most $2c\log n$. Since we aim for such a probability, we may assume that G is a deterministic graph that has this property.

Now, fix any set A of size $0.99n$. Regardless of our choice of A, clearly, every vertex of $V\setminus A$ has at most $2c\log n$ neighbours in $V\setminus A$. Take a random set B from $V\setminus A$ of size $O(n^{0.03})$ and fix a vertex $v\in B$. The probability that no neighbour of v is in B is at least

$$\frac{\binom{0.01n-1-2c\log n}{|B|-1}}{\binom{0.01n-1}{|B|-1}} = 1-O\left(\frac{n^{0.03}\log n}{n}\right) = 1-O\left(n^{-0.97}\log n\right).$$

Hence, the probability that some neighbour of v is in B is $O\left(n^{-0.97}\log n\right)$. Consequently, the union bound over all vertices of B implies that with probability $O\left(n^{-0.94}\log n\right)$ there is at least one edge in the graph induced by B. The proof of the lemma is finished. □

Finally, we are ready to prove the main result of this paper.

Proof of Theorem 1.2. Let $M_- = \frac{n}{2}(\log n - \log\log n)$ and let $M_+ = \frac{n}{2}(\log n + \log\log n)$; recall that a.a.s. $M_- < M < M_+$. First, we will show that a.a.s. $\mathcal{G}(n, M_-)$ contains a good tree T and a small set S of isolated vertices. Next, we will show that between time M_- and M_+ a.a.s. no edge is added between S and light vertices of T. This will finish the proof, since $\mathcal{G}(n, M)$ is connected and so at that point of the process, a.a.s. vertices of S must be adjacent to the spine or heavy vertices of T, which implies that a.a.s. there is a good spanning tree. The result will follow then from Lemma 3.3.

As promised, let $p_- = M_-/\binom{n}{2} = (\log n - \log\log n + o(1))/n$ and consider $G(n, p_-)$ instead of $\mathcal{G}(n, M_-)$. In order to avoid technical problems with events not being independent, we use a classic technique known as *two-round exposure* (known also as *sprinkling* in the percolation literature). The observation is that a random graph $G\in G(n,p)$ can be viewed as a union of two independently generated random graphs $G_1\in G(n,p_1)$ and $G_2\in G(n,p_2)$, provided that $p = p_1+p_2-p_1p_2$ (see, for example, [6, 12] for more information).

Let $p_1 := \log\log\log n/n$ and

$$p_2 := \frac{p_- - p_1}{1-p_1} = \frac{\log n - \log\log n - \log\log\log n + o(1)}{n}.$$

Fix $G_1\in G(n,p_1)$ and $G_2\in G(n,p_2)$, with $V(G_1)=V(G_2)=V$, and view $G\in G(n,p_-)$ as the union of G_1 and G_2. It follows from Lemma 2.1 that with probability $1-o((n\log n)^{-1/2})$, G_1 has a long path $P=(v_1,v_2,\ldots,v_k)$ of length $k=0.99n$ (and thus G has it too). This path will eventually become the spine of the spanning tree.

Now, we expose edges of G_2 in a very specific order dictated by the following algorithm. Initially, A consists of vertices of the path (that is, $A=\{v_1,v_2,\ldots,v_k\}$), $B=\emptyset$, and $C=V\setminus A$. At each step of the process, we take a vertex v from C and expose edges from v to A. If an edge from v to some v_i is found, we change the label of v to u_i, call it heavy, remove v_i from A, and move u_i from C to B. Otherwise (that is, if there is no edge from v to A), we expose edges from v to B. If an edge from v to some heavy vertex u_i is found, we call v light, and remove it from C. Clearly, at the end of

this process we are left with a small (with the desired probability, as we will see soon) set C and a good tree T_1. Let $X = |C|$ be the random variable counting vertices not attached to the tree yet. Since at each step of the process $|A| + |B|$ is precisely $0.99n$, we have

$$\begin{aligned}\mathbb{E}X &= 0.01n\big(1-p_2\big)^{0.99n}\\ &= (0.01+o(1))\exp\big(\log n - 0.99(\log n - \log\log n - \log\log\log n)\big)\\ &= (0.01+o(1))n^{0.01}\big((\log n)(\log\log n)\big)^{0.99}.\end{aligned}$$

Note that X is, in fact, a binomial random variable $\mathrm{Bin}(0.01n, (1-p_2)^{0.99n})$. Hence, it follows from Chernoff's bound (2) that $X = (1+o(1))\mathbb{E}X$ with probability $1 - o((n\log n)^{-1/2})$. Now, let Y be the random variable counting how many vertices *not* on the path are *not* heavy. Arguing as before, since at each step of the process $|A| \geq 0.98n$, we have

$$\mathbb{E}Y \leq 0.01n\big(1-p_2\big)^{0.98n} = (0.01+o(1))n^{0.02}\big((\log n)(\log\log n)\big)^{0.98}.$$

Clearly, $Y \geq X$ and so $\mathbb{E}Y$ is large enough for Chernoff's bound (2) to be applied again to show that $Y = (1+o(1))\mathbb{E}Y$ with probability $1 - o((n\log n)^{-1/2})$.

Our next goal is to attach almost all vertices of C to T_1 in order to form another, slightly larger, good tree T_2. Consider a vertex $v \in C$ and a heavy vertex u_i of T_1. Obvious, yet important, property is that when edges emanating from v were exposed in the previous phase, exactly one vertex from v_i, u_i was considered (recall that when u_i is discovered as a heavy vertex, v_i is removed from A). Hence we may expose these edges and try to attach v to the tree. From the previous argument, since we aim for a statement that holds with probability $1 - o((n\log n)^{-1/2})$, we may assume that the number of heavy vertices is at least $0.01n - n^{0.03}$. For the random variable Z counting vertices still not attached to the path, we get

$$\begin{aligned}\mathbb{E}Z &= (1+o(1))\mathbb{E}X\big(1-p_2\big)^{0.01n-O(n^{0.03})}\\ &= (0.01+o(1))n^{0.01}\big((\log n)(\log\log n)\big)^{0.99}\\ &\qquad\times\exp\big(-0.01(\log n - \log\log n - \log\log\log n)\big)\\ &= (0.01+o(1))\big((\log n)(\log\log n)\big),\end{aligned}$$

and so $Z = (1+o(1))\mathbb{E}Z$ with probability $1 - o((n\log n)^{-1/2})$ by Chernoff's bound (2).

Let us stop for a second and summarize the current situation. We showed that with the desired probability, we have the following structure. There is a good tree T_2 consisting of all but at most $(\log n)(\log\log n)$ vertices that form a set S. T_2 consists of the spine of length $0.99n$, $0.01n - O(n^{0.03})$ heavy vertices and $O(n^{0.03})$ light vertices. Edges between S and the spine and between S and heavy vertices are already exposed and no edge was found. On the other hand, edges within S and between S and light vertices are not exposed yet. However, it is straightforward to see that with the desired probability there is no edge there neither, and so vertices of S are, in fact, isolated in G_2. Indeed, the probability that there is no edge in the graph induced by S and light

vertices is equal to

$$\big(1-p_2\big)^{O(n^{0.06})} = \exp\big(-O(n^{0.06}\log n/n)\big) = 1-o((n\log n)^{-1/2}).$$

Finally, we need to argue that vertices of S are also isolated in G_1. The important observation is that the algorithm we performed that exposed edges in G_2 used only the fact that vertices of $\{v_1, v_2, \ldots, v_{0.99n}\}$ form a long path; no other information about G_1 was ever used. Hence, set S together with light vertices is, from the perspective of the graph G_1, simply a random set of size $O(n^{0.03})$ taken from the set of vertices not on the path. Lemma 3.5 implies that with probability $1-o((n\log n)^{-1/2})$ there is no edge in the graph induced by this set. With the desired property, $G(n,p)$ consists of a good tree T and a small set S of isolated vertices and so, by Lemma 3.4, a.a.s. it is also true in $\mathcal{G}(n, M_-)$.

It remains to show that between time M_- and M_+ a.a.s. no edge is added between S and light vertices of T. Direct computations for the random graph process show that this event holds with probability

$$\begin{aligned}
\binom{\binom{n}{2}-M_- - O(n^{0.06})}{M_+-M_-}\Big/\binom{\binom{n}{2}-M_-}{M_+-M_-} &= \prod_{i=0}^{M_+-M_--1}\frac{\binom{n}{2}-M_- - O(n^{0.06})-i}{\binom{n}{2}-M_- - i}\\
&= \left(1-\frac{O(n^{0.06})}{n^2}\right)^{M_+-M_-}\\
&= 1-\frac{O(n^{0.06}\log\log n)}{n} \to 1
\end{aligned}$$

as $n\to\infty$, since $M_+ - M_- = O(n\log\log n)$. The proof is finished. □

4. Random hypergraphs

First, let us mention that the trivial bound for graphs (1) can easily be generalized to r-uniform hypergraphs and any $2\le k\le r$:

$$\mathcal{AC}_r^k(H) \ge \frac{\binom{|V|}{k}}{|E|\binom{r}{k}} - 1.$$

For $H=(V,E)\in H_r(n,p)$ with $p>(1+\varepsilon)\frac{(r-1)!\log n}{n^{r-1}}$ for some $\varepsilon>0$, we get immediately that a.a.s.

$$\mathcal{AC}_r^k(H) = \Omega\left(\frac{1}{pn^{r-k}}\right),$$

since the expected number of edges in H is $\binom{n}{r}p = \Omega(n\log n)$ and so the concentration follows from Chernoff's bound (2). Hence, the lower bound in Theorem 1.3 holds.

Now we prove the upper bound. In order to do it, we will split Theorem 1.3 into two parts and then, independently, prove each of them.

Theorem 1.3a. *Let $r\in\mathbb{N}\setminus\{1\}$, let $k\in\mathbb{N}$ be such that $2\le k\le r$, and let $\varepsilon>0$ be any real number. Suppose that $p=p(n)=(1+\varepsilon)\frac{(r-1)!\log n}{n^{r-1}}$. For $H\in H_r(n,p)$, a.a.s.*

$$\mathcal{AC}_r^k(H) = O(n^{k-1}) = O\left(\frac{\log n}{pn^{r-k}}\right).$$

Theorem 1.3b. *Let $r \in \mathbb{N} \setminus \{1\}$, let $k \in \mathbb{N}$ be such that $2 \le k \le r$, and let $\omega = \omega(n)$ be any function tending to infinity together with n. Suppose that $p = p(n) = \omega \frac{\log n}{n^{r-1}}$. For $H \in H_r(n,p)$, a.a.s.*

$$\mathcal{AC}_r^k(H) = O\left(\max\left\{\frac{\log n}{pn^{r-k}}, 1\right\}\right).$$

Note that when, say, $p \ge 2k(r-k)!\frac{\log n}{n^{r-k}}$, the expected number of k-tuples that do *not* get acquainted initially (that is, those that are not contained in any hyperedge) is equal to

$$\binom{n}{k}(1-p)^{\binom{n-k}{r-k}} \le \exp\left(k\log n - (1+o(1))2k(r-k)!\frac{\log n}{n^{r-k}} \cdot \frac{n^{r-k}}{(r-k)!}\right) = o(1),$$

and so, by Markov's inequality, a.a.s. every k-tuple gets acquainted immediately and $\mathcal{AC}_r^k(H) = 0$. This is also the reason for taking the maximum of the two values in the statement of the result.

For a given hypergraph $H = (V, E)$, sometimes it will be convenient to think about its *underlying graph* $\tilde{H} = (V, \tilde{E})$ with $\{u, v\} \in \tilde{E}$ if and only if there exists $e \in E$ such that $\{u, v\} \subseteq e$. Clearly, two agents are matched in $\tilde{H}$ if and only if they are matched in H. Therefore, we may always assume that agents walk on the underlying graph of H.

Let $H = (V, E)$ be an r-uniform hypergraph. A *1-factor* of H is a set $F \subseteq E$ such that every vertex of V belongs to exactly one edge in F. Moreover, a *1-factorization* of H is a partition of E into 1-factors, that is, $E = F_1 \cup F_2 \cup \cdots \cup F_\ell$, where each F_i is a 1-factor and $F_i \cap F_j = \emptyset$ for all $1 \le i < j \le \ell$. Denote by K_n^r the *complete* r-uniform hypergraph of order n (that is, each r-tuple is present). We will use the following well-known result of Baranyai [4].

Lemma 4.1 ([4])**.** *If r divides n, then K_n^r is 1-factorizable.*

(Clearly the number of 1-factors is equal to $\binom{n}{r}\frac{r}{n} = \binom{n-1}{r-1}$.) The result on factorization is helpful to deal with loose paths.

Lemma 4.2. *Let $r \in \mathbb{N} \setminus \{1\}$ and let $k \in \mathbb{N}$ be such that $2 \le k \le r$. For a loose r-uniform path P_n on n vertices,*

$$\mathcal{AC}_r^k(P_n) = O(n^{k-1}).$$

Moreover, every agent visits every vertex of the path.

Proof. First, we claim that the number of steps, $f(n)$, needed for agents to be moved to any given permutation of $[n]$, the vertex set of P_n, satisfies $f(n) = O(n)$. For that we will use the fact that the underlying graph $\tilde{P}_n$ contains a classic path (graph) $\bar{P}_n$ of length n as a subgraph. Now, $\bar{P}_n$ can be split into two sub-paths of lengths $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, respectively, and then agents are directed to the corresponding sub-paths that takes $O(n)$ steps by Claim 3.2. We proceed recursively, performing operations on both paths (at the same time) to get that

$$f(n) = O(n) + f(\lceil n/2 \rceil) = O(n).$$

The claim holds.

Let N be the smallest integer such that $N \geq n$ and N is divisible by $k-1$. Clearly, $N - n \leq k-2$. Lemma 4.1 implies that K_N^{k-1} on the vertex set $V(K_N^{k-1}) = [N] \supseteq [n] = V(P_n)$ is 1-factorizable. Let E_i for $1 \leq i \leq \binom{N-1}{k-2}$ be the corresponding 1-factors. Obviously, $|E_i| = \frac{N}{k-1}$ for each i. Let $F_i = \{e \cap [n] : e \in E_i\}$. This way we removed all vertices that are not in P_n and so $F_i \subseteq V(P_n)$ for each $1 \leq i \leq \binom{N-1}{k-2}$ and every k-tuple from $V(P_n)$ is in some F_i. Call every member of F_i a *team*. Moreover, let us mention that, since at most $k-2$ vertices are removed, no team reduces to the empty set and so $|F_i| = \frac{N}{k-1}$ for each i.

Let us focus for a while on F_1. By the above claim we can group all teams of F_1 on P_n in $O(n)$ rounds (using an arbitrary permutation of teams, one team next to the other; see also Remark 4.3). We treat teams of agents as *super-vertices* that form an (abstract) *super-path* of length $\ell = |F_1| = N/(k-1)$. As in the proof of Lemma 3.3, we are going to design the strategy ensuring that within $O(n)$ rounds every team of agents gets acquainted with every other agent and, moreover, that every agent visits every vertex of P. Let u_i be the i-th super-vertex corresponding to the i-th team. For $1 \leq i \leq \ell - 1$, let $e_i = u_i u_{i+1}$. The strategy is divided into 2ℓ phases, each consisting of $O(1)$ rounds. (The number of rounds is a function of k.) On odd-numbered phases, swap teams agents on all odd-indexed edges. On even-numbered phases, swap teams agents on all even-indexed edges. This guarantees that every pair of teams eventually meets, as discussed in the proof of Lemma 3.3. (Note that super-vertices on the super-path in the current argument correspond to vertices on the path in the argument used in Lemma 3.3.) The only difference is that this time swapping teams usually cannot be done in one round but clearly can be done in $O(1)$ rounds. (For example, the first agent from the second team swaps with each agent from the first team to move to the other side of the underlying path $\tilde{P}_n$. This takes at most $k-1$ swaps. Then the next agent from the second team repeats the same, and after at most $(k-1)^2$ rounds we are done; the number of rounds would be precisely $(k-1)^2$ if swaps are done in the 'bubble sort' manner which is fine for our purpose but clearly sub-optimal.) Moreover, this procedure can be done in such a way that each team meets each agent from the passing team on some hyperedge of P_n. This might require additional $O(1)$ rounds but can easily be done. (Indeed, note that the set of vertices corresponding to the two super-vertices used must contain a set of k vertices all of which belong to the same hyperedge of P_n. So, for example, the first team consisting of at most $k-1$ agents can go to such set and let every agent from the second team to pass them, meeting the whole team at some point, and come back to the initial position. The teams swap roles and finally do the swap.)

After $O(n)$ steps, each team from F_1 consisting of $k-1$ agents, is k-acquainted with each agent. Repeating this process for each F_i yields the desired strategy for the total of $\binom{N-1}{k-2} \cdot O(n) = O(n^{k-1})$ rounds. □

Remark 4.3. As we already mentioned, in the previous proof, the order in which teams are distributed for a given 1-factor was not important. One can fix an arbitrary order or, say, a random one that will turn out to be a convenient way of dealing with some small technical issue in the proof of Theorem 1.3b.

It remains to prove the two theorems.

Proof of Theorem 1.3a. As in the proof of Theorem 1.2, we use two-round exposure technique: $H \in H_r(n,p)$ can be viewed as the union of two independently generated random hypergraphs $H_1 \in H_r(n,p_1)$ and $H_2 \in H_r(n,p_2)$, with $p_1 := \frac{\varepsilon}{2} \cdot \frac{(r-1)!\log n}{n^{r-1}}$ and

$$p_2 := \frac{p-p_1}{1-p_1} \geq p - p_1 = \left(1+\frac{\varepsilon}{2}\right)\frac{(r-1)!\log n}{n^{r-1}}.$$

It follows from Lemma 2.1 that a.a.s. H_1 has a long loose path P of length δn (and thus H has it, too), where $\delta \in (0,1)$ can be made arbitrarily close to 1, as needed.

Now, we will use H_2 to show that a.a.s. every vertex not on the path P belongs to at least one edge of H_2 (and thus H, too) that intersects with P. Indeed, for a given vertex v not on the path P, the number of r-tuples containing v and at least one vertex from P is equal to

$$\begin{aligned} s &= \binom{n-1}{r-1} - \binom{(1-\delta)n-1}{r-1} = (1+o(1))\left(\frac{n^{r-1}}{(r-1)!} - \frac{((1-\delta)n)^{r-1}}{(r-1)!}\right) \\ &= (1+o(1))\frac{n^{r-1}}{(r-1)!}\left(1-(1-\delta)^{r-1}\right), \end{aligned}$$

and so the probability that none of them occurs as an edge in H_2 is

$$(1-p_2)^s \leq \exp\left(-(1+o(1))\left(1+\frac{\varepsilon}{2}\right)\frac{(r-1)!\log n}{n^{r-1}} \cdot \frac{n^{r-1}}{(r-1)!}\left(1-(1-\delta)^{r-1}\right)\right) = o(n^{-1}),$$

after taking δ close enough to 1. The claim holds by the union bound.

The rest of the proof is straightforward. We partition all agents into $(k+1)$ groups, $A_1, A_2, \ldots, A_{k+1}$, each consisting of at least $\lfloor n/(k+1) \rfloor$ agents. Since δ can be arbitrarily close to 1, we may assume that $\lceil n/(k+1) \rceil > (1-\delta)n$. For every $i \in [k+1]$, we route *all* agents from all groups but A_i and some agents from A_i to P, and after $O(n^{k-1})$ rounds they get acquainted by Lemma 4.2. Since each k-tuple of agents intersects at most k groups, it must get acquainted eventually, and the proof is complete. □

Proof of Theorem 1.3b. First we assume that $\omega = \omega(n) \leq n^{k-1}$ and $r-1$ divides $n-1$. As usual, we consider two independently generated random hypergraphs $H_1 \in H_r(n,p_1)$ and $H_2 \in H_r(n,p_2)$, with $p_1 := \frac{\omega}{2} \cdot \frac{\log n}{n^{r-1}}$ and $p_2 > p - p_1 = p_1$. It follows from Lemma 2.2 that a.a.s. H_1 has a Hamiltonian path $P = (v_1, v_2, \ldots, v_n)$.

Let C be a large constant that will be determined soon. We cut P into $(\omega/C)^{1/(k-1)}$ loose sub-paths, each of length $\ell := (C/\omega)^{1/(k-1)}n$; there are $r-2$ vertices between each pair of consecutive sub-paths that are not assigned to any sub-path for a total of at most $r\ell = r \cdot (\omega/C)^{1/(k-1)} \leq rnC^{-1/(k-1)}$ vertices, since $\omega \leq n^{k-1}$. This can be made smaller than, say, $\lceil n/(k+1) \rceil$, provided that C is large enough. We will call agents occupying these vertices to be *passive*; agents occupying sub-paths will be called *active*. Active agents will be partitioned into units, with respect to the sub-path they occupy. Every unit performs (independently and simultaneously) the strategy from Lemma 4.2 on their own sub-path. After $s := O(Cn^{k-1}/\omega)$ rounds, each k-tuple of agents from the same unit will be k-acquainted.

Now, we will show that a.a.s. every k-tuple of active agents, not all from the same unit, gets acquainted. Let us focus on one such k-tuple, say f. If one starts from a random order in the strategy from Lemma 4.2 (see Remark 4.3), then it is clear that with probability $1 - o(n^k)$ agents from f visit at least $s/2 = Cn^{k-1}/(2\omega)$ different k-tuples of vertices. Hence, by considering vertices that are not on paths agents from f walk on, we get that with probability $1 - o(n^k)$, the number of distinct r-tuples visited by agents from f is at least

$$t := \frac{s}{2}\binom{n-k\ell}{r-k} \geq \frac{Cn^{k-1}}{2\omega} \cdot \frac{(0.9n)^{r-k}}{(r-k)!} = \frac{C(0.9)^{r-k}}{2(r-k)!} \cdot \frac{n^{r-1}}{\omega}.$$

Considering only those edges in H_2, the probability that agents from f never got acquainted is at most

$$(1-p_2)^t \leq \exp\left(-\frac{\omega \log n}{n^{r-1}} \cdot \frac{C(0.9)^{r-k}}{2(r-k)!} \cdot \frac{n^{r-1}}{\omega}\right) = \exp\left(-\frac{C(0.9)^{r-k}}{2(r-k)!}\log n\right) = o(n^{-k}),$$

for C sufficiently large. Since there are at most $\binom{n}{k} = O(n^k)$ k-tuples of agents, a.a.s. all active agents got acquainted.

It remains to deal with passive agents which can be done the same way as in the proof of Theorem 1.3a. We partition all agents into $(k+1)$ groups, and shuffle them so that all get acquainted. Moreover, if $r-1$ does not divide $n-1$, we may take N to be the largest integer such that $N \leq n$ and $r-1$ divides $N-1$, and deal with a loose path on N vertices. Clearly, a.a.s. each of the remaining $O(1)$ vertices belong to at least one hyperedge intersecting with P. We may treat these vertices as passive ones and proceed as before.

Finally, suppose that $\omega = \omega(n) > n^{k-1}$. Since $\mathcal{AC}_r^k(H_r(n,p))$ is a decreasing function of p,

$$\mathcal{AC}_r^k(H_r(n,p)) \leq \mathcal{AC}_r^k(H_r(n,(\log n)/n^{r-k})) = O(1),$$

by the previous case and the theorem is finished. □

5. Concluding remarks

In this paper, we showed that as soon as the random graph process $\mathcal{G}(n,m)$ becomes connected its acquaintance time is linear, which implies that for $G \in G(n,p)$ the following property holds a.a.s.: if G is connected, then $\mathcal{AC}(G) = O(\log n/p)$. This confirms the conjecture from [5] in the strongest possible sense. On the other hand, the trivial lower bound (1) is off by a factor of $\log n$, namely, for $G \in G(n,p)$ the following property holds a.a.s.: if G is connected, then $\mathcal{AC}(G) = \Omega(1/p)$. It is worth to mentioning that it was shown in [14] that for dense random graphs (that is, graphs with average degree at least $n^{1/2+\varepsilon}$ for some $\varepsilon > 0$) the upper bound gives the right order of magnitude. It would be interesting to strengthen these bounds and investigate the order of $\mathcal{AC}(G)$ for sparse random graphs.

Of course, an analogous question could be asked (and probably should) for hypergraphs. It would be desired to improve the bounds in Theorem 1.3. Moreover, in Theorem 1.3, it is assumed that p is by a multiplicative factor of $(1+\varepsilon)$ larger than the threshold for connectivity ($\varepsilon > 0$ is any constant, arbitrarily small). It remains to

investigate the behaviour of $\mathcal{AC}_r^k(H)$ in the critical window, preferably by investigating the random hypergraph process as it was done for graphs in Theorem 1.2.

It might also be of some interest to study the acquaintance time of other models of random graphs such as, for example, power-law graphs, preferential attachments graphs, or geometric graphs. The latter was recently considered in [16]. Moreover, let us mention that there are some number of interesting open problems for deterministic graphs, see, for example [5].

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Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA
E-mail address: `andrzej.dudek@wmich.edu`

Department of Mathematics, Ryerson University, Toronto, ON, Canada
E-mail address: `pralat@ryerson.ca`