VARIANTS OF THE GAME OF COPS AND ROBBERS

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Abstract. In this extended abstract, we survey the game of Cops and Robbers and a few variants with a special focus on the game of Containment, in which cops move from edge to adjacent edge, the robber moves from vertex to adjacent vertex (but cannot move along an edge occupied by a cop). The cops win by "containing" the robber, that is, by occupying all edges incident with a vertex occupied by the robber.

1. Introduction

The game of Cops and Robbers is usually studied in the context of the *cop number*, the minimum number of cops needed to ensure a winning strategy. The cop number is often challenging to analyze; establishing upper bounds for this parameter is the focus of Meyniel's conjecture that the cop number of a connected n-vertex graph is $O(\sqrt{n})$. For additional background on Cops and Robbers and Meyniel's conjecture, see the book [7].

A number of variants of Cops and Robbers have been studied. For example, we may allow a cop to capture the robber from a distance k, where k is a non-negative integer [5], play on edges [9], allow one or both players to move with different speeds [1, 10] or to teleport, allow the robber to capture the cops [6], make the robber invisible or drunk [11, 12], or allow at most one cop to move in any given round [15, 2, 3]. See Chapter 8 of [7] for a non-comprehensive survey of variants of Cops and Robbers.

In this extended abstract, we present a few results for the variant of the game of Cops and Robbers, called *Containment*, introduced recently by Komarov and Mackey [13]. In this version, cops move from edge to adjacent edge, the robber moves as in the classic game, from vertex to adjacent vertex (but cannot move along an edge occupied by a cop). Formally, the game is played on a finite, simple, and undetected graph. There are two players, a set of *cops* and a single *robber*. The game is played over a sequence of discrete time-steps or *turns*, with the cops going first on turn 0 and then playing on alternate time-steps. A *round* of the game is a cop move together with the subsequent robber move. The cops occupy edges and the robber occupies vertices; for simplicity, we often identify the player with the vertex/edge they occupy. When the robber is ready to move in a round, she can move to a neighbouring vertex but cannot move along an edge occupied by a cop, cops can move to an edge that is incident to their current location. Players can always *pass*, that is, remain on their own vertices/edges. Observe that any subset of cops may move in a given round. The cops win if after some finite number of rounds, all edges incident with the robber are occupied by cops. This is called a *capture*. The robber wins if she can evade capture indefinitely. A *winning strategy for the cops* is a set of rules that if followed, result in a win for the cops. A *winning*

1991 *Mathematics Subject Classification.* 05C57, 05C80.
Key words and phrases. Containment, Cops and Robbers, vertex-pursuit games, random graphs.
Supported by grants from NSERC and Ryerson.

strategy for the robber is defined analogously. As stated earlier, the original game of *Cops and Robbers* is defined almost exactly as this one, with the exception that all players occupy vertices.

If we place a cop at each edge, then the cops are guaranteed to win. Therefore, the minimum number of cops required to win in a graph G is a well-defined positive integer, named the *containability number* of the graph G. Following the notation introduced in [13], we write $\xi(G)$ for the containability number of a graph G and $c(G)$ for the original *cop-number* of G.

In [13], Komarov and Mackey proved that for every graph G,

$$c(G) \le \xi(G) \le \gamma(G)\Delta(G),$$

where $\gamma(G)$ and $\Delta(G)$ are the domination number and the maximum degree of G, respectively. It was conjectured that the upper bound can be strengthened and, in fact, the following holds.

Conjecture 1.1 ([13]). *For every graph G, $\xi(G) \le c(G)\Delta(G)$.*

Observe that, trivially, $c(G) \le \gamma(G)$ so this would imply the previous result. This seems to be the main question for this variant of the game at the moment. By investigating expansion properties, we provide asymptotically almost sure bounds on the containability number of binomial random graphs $\mathcal{G}(n,p)$ for a wide range of $p = p(n)$, proving that the conjecture holds for some ranges of p (or holds up to a constant or an $O(\log n)$ multiplicative factors for some other ranges of p). However, before we state the result, let us introduce the probability space we deal with and mention a few results for the classic cop-number that will be needed to examine the conjecture (since the corresponding upper bound is a function of the cop number).

The *random graph* $\mathcal{G}(n,p)$ consists of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all graphs with vertex set $\{1, 2, \dots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$,

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

This space may be viewed as the set of outcomes of $\binom{n}{2}$ independent coin flips, one for each pair (u,v) of vertices, where the probability of success (that is, adding edge uv) is p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity. All asymptotics throughout are as $n \to \infty$ (we emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded). We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity.

Let us now briefly describe some known results on the (classic) cop-number of $\mathcal{G}(n,p)$. Bonato, Wang, and the author of this paper investigated such games in $\mathcal{G}(n,p)$ random graphs and in generalizations used to model complex networks with power-law degree distributions (see [8]). From their results it follows that if $2\log n/\sqrt{n} \le p < 1-\varepsilon$ for some $\varepsilon > 0$, then a.a.s. we have that

$$c(\mathcal{G}(n,p)) = \Theta(\log n/p),$$

so Meyniel's conjecture holds a.a.s. for such p. In fact, for $p = n^{-o(1)}$ we have that a.a.s. $c(\mathcal{G}(n,p)) = (1+o(1))\log_{1/(1-p)} n$. A simple argument using dominating sets shows that

Meyniel's conjecture also holds a.a.s. if p tends to 1 as n goes to infinity (see [17] for this and stronger results). Bollobás, Kun and Leader [4] showed that if $p(n) \geq 2.1 \log n/n$, then a.a.s.

$$\frac{1}{(pn)^2} n^{1/2 - 9/(2 \log\log(pn))} \leq c(\mathcal{G}(n,p)) \leq 160000\sqrt{n} \log n\,.$$

From these results, if $np \geq 2.1 \log n$ and either $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$, then a.a.s. $c(\mathcal{G}(n,p)) = n^{1/2+o(1)}$. Somewhat surprisingly, between these values it was shown by Łuczak and the author of this paper [14] that the cop number has more complicated behaviour. It follows that a.a.s. $\log_n c(\mathcal{G}(n, n^{x-1}))$ is asymptotic to the function $f(x)$ shown in Figure 1 (denoted in blue).

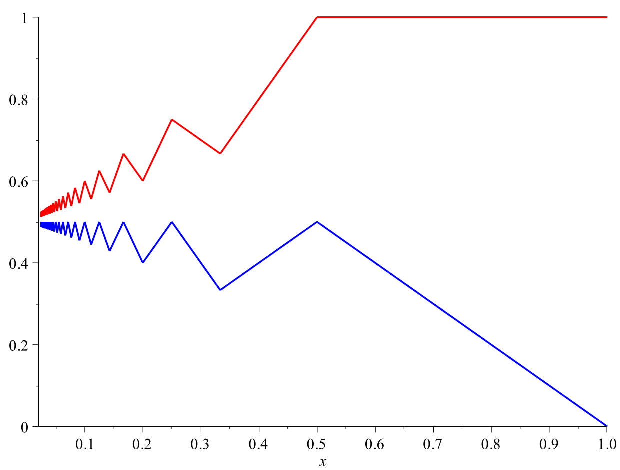

Figure 1. The "zigzag" functions representing the ordinary cop number (blue) and the containability number (red).

The above result shows that Meyniel's conjecture holds a.a.s. for random graphs except perhaps when $np = n^{1/(2k)+o(1)}$ for some $k \in \mathbb{N}$, or when $np = n^{o(1)}$. The author of this paper and Wormald showed recently that the conjecture holds a.a.s. in $\mathcal{G}(n,p)$ [18] as well as in random d-regular graphs [19].

Finally, we are able to state the result that was recently obtained in [16].

Theorem 1.2 ([16]). *Let $0 < \alpha < 1$ and $d = d(n) = np = n^{\alpha+o(1)}$.*

(i) *If $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ for some integer $j \geq 1$, then a.a.s.*

$$\xi(\mathcal{G}(n,p)) = \Theta(d^{j+1}) = \Theta(c(\mathcal{G}(n,p)) \cdot \Delta(\mathcal{G}(n,p)))\,.$$

Hence, a.a.s. Conjecture 1.1 holds (up to a multiplicative constant factor).

(ii) *If $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ for some integer $j \geq 2$, then a.a.s.*

$$\begin{aligned} \xi(\mathcal{G}(n,p)) &= \Omega\left(\frac{n}{d^{j-1}}\right), \text{ and} \\ \xi(\mathcal{G}(n,p)) &= O\left(\frac{n \log n}{d^{j-1}}\right) = O(c(\mathcal{G}(n,p)) \cdot \Delta(\mathcal{G}(n,p)) \cdot \log n)\,. \end{aligned}$$

Hence, a.a.s. Conjecture 1.1 holds (up to a multiplicative $O(\log n)$ factor).

(iii) *If $1/2 < \alpha < 1$, then a.a.s.*

$$\xi(\mathcal{G}(n,p)) = \Theta(n) = \Theta(c(\mathcal{G}(n,p)) \cdot \Delta(\mathcal{G}(n,p))/\log n) \le c(\mathcal{G}(n,p)) \cdot \Delta(\mathcal{G}(n,p).$$

Hence, a.a.s. Conjecture 1.1 holds.

It follows that a.a.s. $\log_n \xi(\mathcal{G}(n, n^{x-1}))$ is asymptotic to the function $g(x)$ shown in Figure 1 (denoted in red). The fact the conjecture holds is associated with the observation that $g(x) - f(x) = x$, which is equivalent to saying that a.a.s. the ratio $\xi(\mathcal{G}(n,p))/c(\mathcal{G}(n,p)) = dn^{o(1)} = \Delta(\mathcal{G}(n,p)) \cdot n^{o(1)}$. Moreover, let us mention that Theorem 1.2 implies that the conjecture is best possible (again, up to a constant or an $O(\log n)$ multiplicative factors for corresponding ranges of p).

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