# Chasing robbers on percolated random geometric graphs

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**Abstract**

In this paper, we study the vertex pursuit game of *Cops and Robbers*, in which cops try to capture a robber on the vertices of a graph. The minimum number of cops required to win on a given graph $G$ is called the cop number of $G$. We focus on $\mathcal{G}(n, r, p)$, a percolated random geometric graph in which $n$ vertices are chosen uniformly at random and independently from $[0, 1]^2$. Two vertices are adjacent with probability $p$ if the Euclidean distance between them is at most $r$. If the distance is bigger then $r$ then they are never adjacent. We present asymptotic results for the game of Cops and Robber played on $\mathcal{G}(n, r, p)$ for a wide range of $p = p(n)$ and $r = r(n)$.

## 1 Introduction and Results

The game of *Cops and Robbers*, introduced independently by Nowakowski and Winkler [13] and Quilliot [19] more than thirty years ago, is played on a fixed graph $G$. We will always assume that $G$ is undirected, simple, and finite. There are two players, a set of $k$ *cops*, where $k \geq 1$ is a fixed integer, and the *robber*. The cops begin the game by occupying any set of $k$ vertices (in fact, for a connected $G$, their initial position does not matter). The robber then chooses a vertex. In each subsequent round, the cops first move and then the robber moves. The players use edges to move from vertex to vertex. More than one cop is allowed to occupy a vertex, and the players may remain on their current positions. The players know each others current locations. The cops win and the game ends if at least one of the cops eventually occupies the same vertex as the robber; otherwise, that is, if the robber can avoid this indefinitely, she wins. As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, which is the minimum number of cops needed to win on $G$. The cop number was introduced by Aigner and Fromme [1] who proved (among other things) that if $G$ is planar, then $c(G) \leq 3$. The most important open problem in this area is Meyniel's conjecture (communicated by Frankl [8]). It states that $c(n) = O(\sqrt{n})$, where $c(n)$ is the maximum of $c(G)$ over all $n$-vertex connected graphs. If true, the estimate is best possible as one can construct a graph based on the finite projective plane with the cop number of order at least $\Omega(\sqrt{n})$. Up until recently, the best known upper bound of $O(n \log \log n / \log n)$ was given in [8]. This was improved to $c(n) = O(n/ \log n)$ in [7]. Today we know that the cop number is at most $n 2^{-(1+o(1))\sqrt{\log_2 n}}$ (which is still $n^{1-o(1)}$) for any connected graph on $n$ vertices (a result obtained independently by Lu and Peng [11] and

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Scott and Sudakov [20], see also [9] for some extensions). If one looks for counterexamples for Meyniel's conjecture it is natural to study first the cop number of random graphs. Recent years have witnessed significant interest in the study of random graphs from that perspective [4, 6, 12, 16] confirming that, in fact, Meyniel's conjecture holds asymptotically almost surely for binomial random graphs [18] as well as for random $d$-regular graphs [17]. For more results on vertex pursuit games such as *Cops and Robbers*, the reader is directed to the monograph [5].

In this paper, we consider a *percolated random geometric graph* $\mathcal{G}(n, r, p)$ which is defined as a random graph with vertex set $V = \{X_1, X_2, \ldots, X_n\}$ in which the $X_i$-s are chosen uniformly at random and independently from the unit square $[0, 1]^2$, and for each pair of vertices within Euclidean distance at most $r$ we flip a biased coin with success probability $p$ to determine whether there is an edge (independently for each such a pair, and pairs at distance bigger than $r$ never share an edge). In particular, for $p = 1$ we get a *(classic) random geometric graph* $\mathcal{G}(n, r)$—see, for example, the monograph [14]. Percolated random geometric graphs were recently studied by Penrose [15] under the name *soft random geometric graphs.* In [15] the connectivity of percolated random geometric graphs was considered, and in particular it was shown that the probability of being connected is governed by the probability of having no isolated vertices, much like in the case of the Erdős-Rényi model or the (unpercolated) classical random geometric graph model.

As typical in random graph theory, in this paper we shall focus on asymptotic properties of $\mathcal{G}(n, r, p)$ as $n \to \infty$, where $p = p(n)$ and $r = r(n)$ may and usually do depend on $n$. We say that an event in a probability space holds *asymptotically almost surely* (*a.a.s.*) if its probability tends to one as $n$ goes to infinity.

The following result for classic random geometric graphs was obtained independently in [3] and in [2].

**Theorem 1.1 ([3, 2])** *There exists an absolute constant $c > 0$ so that if $r^5 > c\frac{\log n}{n}$ then a.a.s. $c(\mathcal{G}(n, r)) = 1$.*

In [3], the known necessary and sufficient condition for a graph to be cop-win (see [13] for more details) is used; that is, it is shown that the random geometric graph is what is called dismantlable a.a.s. The proof in [2] is quite different, provides a tight $O(1/r^2)$ bound for the number of rounds required to catch the robber, and can be generalized to higher dimensions. In the proof an explicit strategy for the cop is introduced and it is shown that it is a winning one a.a.s. Essentially the same proof also gives a generalization of the result to higher dimensions. In [3] it was also shown that every (not necessarily random) connected geometric graph has cop number at most nine, that a.a.s. $c(\mathcal{G}(n, r)) \leq 2$ if $r^4 > c \log n/n$ for some absolute constant $c$, and that there are sequences $r$ for which $\mathcal{G}(n, r)$ is a.a.s. connected while its cop-number is strictly larger than one.

In this paper, we consider the cop number of percolated random geometric graphs. In particular, we will prove the following result.

**Theorem 1.2** *For every $\varepsilon > 0$, and functions $p = p(n)$ and $r = r(n)$ so that $p^2r^2 \geq n^{-1+\varepsilon}$ and $p \leq 1 - \varepsilon$ we have that a.a.s. $c(\mathcal{G}(n, r, p)) = \Theta\left(\frac{\log n}{p}\right)$.*

We find this result quite surprising, since the asymptotics of the cop number for a large range of the parameters does not depend on $r$ but only on $p$. We conjecture that, under the conditions of our theorem, a.a.s. the cop number is $(1 + o(1)) \log_{1/(1-p)} n$.

## 2 Proofs

For $0 < p \leq 1 - \varepsilon$ for some $\varepsilon > 0$, it is convenient to define

$$\mathbb{L} = \mathbb{L}(n) := \log_{1/(1-p)} n,$$

and to state our intermediate results in terms of $\mathbb{L}$. Note that $\mathbb{L} = \Theta\left(\frac{\log n}{p}\right)$.

We will use the following version of *Chernoff bound*. (For more details, see, for example, [10].) Suppose that $X \in \mathrm{Bin}(n, p)$ is a binomial random variable with expectation $\mu = np$. If $0 < \delta < 1$, then

$$\mathbb{P}[X < (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2}\right),$$

and if $\delta > 0$,

$$\mathbb{P}[X > (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2+\delta}\right).$$

The lower and upper bounds are proved separately in the following two subsections.

### 2.1 Lower bound of Theorem 1.2

For the proof of the lower bound, we employ the following adjacency property that was used for dense binomial random graphs [6]. For a fixed $k > 0$ an integer, we say that $G$ is $(1, k)$-*existentially closed* (or $(1, k)$-*e.c.*) if for each $k$-set $S$ of vertices of $G$ and vertex $u \notin S$, there is a vertex $z \notin S \cup \{u\}$ not joined to a vertex in $S$ and joined to $u$. If $G$ is $(1, k)$-*e.c.*, then $c(G) > k$. (The robber may use the property to construct a wining strategy against $k$ cops; she escapes to a vertex not joined to any vertex occupied by a cop.) Hence, to prove the lower bound in Theorem 1.2 it suffices to prove the following.

**Lemma 2.1** *Writing $k := \lfloor \varepsilon \mathbb{L}/2 \rfloor$ – where $\varepsilon > 0$ is as provided by the conditions of Theorem 1.2 – we have that, a.a.s., $\mathcal{G}(n, r, p)$ is $(1, k)$-e.c. In particular, a.a.s. $c(G(n, r, p)) > k$.*

**Proof:** Let $s(u)$ be the number of vertices within Euclidean distance $r$ from $u$. It follows easily from Chernoff bound that there exists a function $t = t(n) = \Omega(r^2 n)$ such that a.a.s. for every vertex $u \in V(G)$, $s(u) \geq t$. Since our goal is to show a result that holds a.a.s. we may assume that this property holds deterministically. More precisely, we think of revealing the graph in two stages. In the first stage, we reveal only the locations of the points, in the second we reveal the relevant coin flips. In the remainder of the proof all mention of probability, expectation, etc., will be with respect to the situation where we have passed the first stage and it turned out that $s(u) \geq t$ for all $u \in V$. In other words, the only randomness we consider is in the coin flips deciding which pairs of points at Euclidean distance at most $r$ will become the edges of our graph.

Fix $S$, a $k$-subset of vertices and a vertex $u$ not in $S$. For a vertex $x \in V(G) \setminus (S \cup \{u\})$ that is within distance $r$ of $u$, the probability that $x$ is joined to $u$ and to no vertex of $S$ is at least $p(1-p)^k$ (note that this is a lower bound only, since $y \in S$ is adjacent to $x$ with probability $p$, provided that the distance between them is at most $r$; otherwise, they are not adjacent). Since edges are chosen independently, the probability that no suitable vertex can be found for this particular $S$ and $u$ is at most

$$(1 - p(1-p)^k)^{t-k-1} = (1 - p(1-p)^k)^{\Omega(r^2 n)}.$$

Let $X$ be the random variable counting the number of $S$ and $u$ for which no suitable $x$ can be found. (Remember that this is after we have revealed the locations of the points.) We then have that

$$\begin{aligned}
\mathbb{E}(X) &\leq n\binom{n}{k}\left(1-p(1-p)^k\right)^{\Omega(r^2n)} \\
&\leq n^{k+1}\exp[-\Omega(p(1-p)^k nr^2)] \\
&= \exp\left[(k+1)\log n - \Omega(n^{-\varepsilon/2}\cdot pnr^2)\right] \\
&\leq \exp\left[O(\log^2 n/p) - \Omega(n^{-\varepsilon/2}\cdot pnr^2)\right] \\
&= o(1),
\end{aligned}$$

where in the third line we have used the definition of $k$ and the last inequality follows from $p^2r^2 \geq n^{-1+\varepsilon}$ (which implies that $\log^2 n/p \ll n^{-\varepsilon/2}\cdot pnr^2$). This concludes the proof of the lemma. ■

## 2.2 Upper bound of Theorem 1.2

In this section we show that, a.a.s., $21000\mathbb{L}$ cops suffice to catch the robber. Before presenting a winning strategy of the cops, we give some preparatory lemmas.

### 2.2.1 Preliminaries

We say that a set of vertices $A \subseteq V$ *dominates* another set of vertices $B \subseteq V$ if every vertex of $B$ is adjacent to some vertex of $A$. Throughout this paper we will denote by $B(x,s) := \{y \in \mathbb{R}^2 : \|x-y\| \leq s\}$ the ball of radius $s$ around $x$.

**Lemma 2.2** *A.a.s., for every $v, w \in V$ with $\|v-w\| \leq 0.99\cdot r$, there is a subset $A \subseteq N(v)$ with $|A| \leq 1000\mathbb{L}$ that dominates $\{w\}\cup N(w)$.*

**Proof:** We will consider the number of "bad" (ordered) pairs $(v,w) \in V^2$ such that $\|v-w\| \leq 0.99\cdot r$, yet no set $A$ as required by the lemma exists. We will compute the probability that $(X_1, X_2)$ form such a bad pair. To do this, we reveal the graph in three stages. In the first stage we reveal $V$ (the positions of the points). In the second stage, we reveal all edges that have $X_1$ as an endpoint (i.e. all coin flips that determine these edges). In the third stage, we reveal all other edges (coin flips).

Let us condition on the event that $\|X_1 - X_2\| \leq 0.99\cdot r$. (Note this does not affect the locations of the other points nor the status of any of the coin flips). We now define, for $i,j \in \{-1,+1\}$:

$$\begin{aligned}
B_{i,j} &:= B(X_2 + i(r/10^{10})e_1 + j(r/10^{10})e_2, r/10^{10}), \\
U_{i,j} &:= N(X_1)\cap B_{i,j}.
\end{aligned}$$

(Here, of course, $e_1 = (1,0)$ and $e_2 = (0,1)$. See Figure 1 for a depiction.) The $B_{i,j}$ have been chosen so that, no matter where in the unit square $X_2$ falls, for every $z \in B(X_2,r)\cap[0,1]^2$ there is at least one pair $(i,j) \in \{-1,1\}^2$ such that $B_{i,j} \subseteq B(z,r)\cap[0,1]^2$.

Observe that, conditioning on the event that the position of $X_2$ is such that $B_{i,j} \subseteq [0,1]^2$, we have

$$|U_{i,j}| \stackrel{d}{=} \mathrm{Bin}(n-2, p\pi(r/10^{10})^2).$$

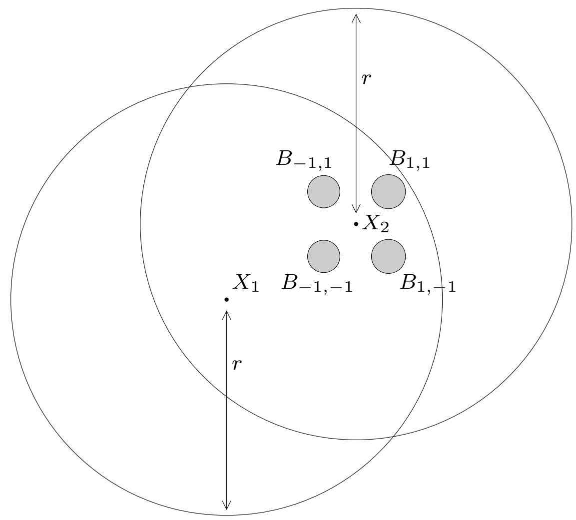


Figure 1: The definition of the $U_{i,j}$. (Not to scale.)

In particular, $\mathbb{E}|U_{i,j}| = \Omega(pnr^2) = \Omega(n^\varepsilon/p) \gg \mathbb{L}$. Using Chernoff bound, it follows that

$$\mathbb{P}(|U_{i,j}| < \mathbb{E}|U_{i,j}|/2) \le \exp[-\Omega(n^\varepsilon)].$$

Note that, to find the $U_{i,j}$ we have to reveal the first two stages, but we do not need to reveal the coin flips corresponding to potential edges not involving $X_1$. Assuming that in the first two stages we managed to find $U_{i,j}$'s of size at least half of the expected size, we can now fix, for each $i, j \in \{-1, 1\}$ with $B_{i,j} \subseteq [0,1]^2$, an arbitrary subset $A_{i,j} \subseteq U_{i,j}$ with $|A_{i,j}| = 250\mathbb{L}$. We let $A$ be the union of these $A_{i,j}$'s. Since each $z \in B(X_2, r) \cap [0,1]^2$ satisfies $A_{i,j} \subseteq B(z, r)$ for at least one pair $(i, j) \in \{-1, 1\}^2$, the probability that there is a vertex $X_j \in N(X_2) \cup \{X_2\}$ not connected by an edge to any vertex of $A$ is at most $n(1-p)^{250\mathbb{L}} = n^{-249}$. It follows that

$$\mathbb{P}((X_1, X_2) \text{ is a bad pair}) \le 4e^{-\Omega(n^\varepsilon)} + n^{-249} \le 2n^{-249},$$

the last inequality holding for sufficiently large $n$. This shows that the expected number of bad pairs is at most $\binom{n}{2} 2n^{-249} = o(1)$. The lemma follows by Markov's inequality. ■

**Lemma 2.3** *A.a.s., for every $v \in V$ and every $z \in B(v, r) \cap [0,1]^2$ there is a vertex $w \in N(v) \cap B(z, r/1000)$.*

**Proof:** We dissect $[0,1]^2$ into squares of side $s := 1/\lceil \frac{10^{10}}{r} \rceil$ (note $s \le r/10^{10}$ and $s = \Theta(r)$). Observe that if $v, z \in [0,1]^2$ with $\|v - z\| \le r$ then there is at least one square of our dissection contained in $B(v, r) \cap B(z, r/1000)$. It thus suffices to count the number $Z$ of "bad pairs" consisting of a vertex $v$ and a square $S$ of the dissection contained in $B(v, r)$ such that $N(v) \cap S = \emptyset$, and to show this number is zero a.a.s. Note that the number of squares is $O(1/r^2) = O(n)$. Hence we have

$$\begin{aligned} \mathbb{E}Z &= O(n^2) \cdot (1 - ps^2)^{n-1} = O(n^2) \cdot \exp[-\Omega(pnr^2)] \\ &= \exp[O(\log n) - \Omega(n^\varepsilon)] = o(1), \end{aligned}$$

and the proof of the lemma is finished by Markov's inequality. ■

**Lemma 2.4** *A.a.s., for every $v, w \in V$ with $\|v - w\| \leq 1.99r$ there is a vertex $u$ such that $uv, uw \in E$ and $\|u - (v+w)/2\| \leq r/1000$.*

**Proof:** We use the same dissection into small squares of side $s := 1/\lceil \frac{10^{10}}{r} \rceil$ as in the proof of the previous lemma. Note that if $v, w \in [0,1]^2$ then $B((v+w)/2, r/1000)$ contains at least one square of the dissection. It thus suffices to count the number $Z$ of "bad triples" consisting of two vertices $v \neq w$ at distance at most $1.99r$ and one square $S$ of the dissection that is contained in $B((v+w)/2, r/1000)$, such that $N(v) \cap N(w) \cap S = \emptyset$. We have

$$\begin{aligned} \mathbb{E}Z &\leq O(n^3) \cdot (1 - p^2 s^2)^{n-2} = O(n^3) \cdot \exp[-\Omega(p^2 n r^2)] \\ &= \exp[O(\log n) - \Omega(n^{\varepsilon})] = o(1), \end{aligned}$$

proving the lemma. ■

The (easy) proof of the next, standard and elementary, observation is left to the reader.

**Lemma 2.5** *Suppose that $x_1, x_2, y_1, y_2 \in \mathbb{R}^2$ are such that $\|x_1 - x_2\|, \|y_1 - y_2\| \leq r$ and the line segments $[x_1, x_2], [y_1, y_2]$ cross. Then $\|x_i - y_j\| \leq r/\sqrt{2}$ for at least one pair $(i, j) \in \{1, 2\}^2$.*

We say that a cop $C$ *controls* a path $P$ in a graph $G$ if whenever the robber steps onto $P$, then she either steps onto $C$ or is caught by $C$ on her responding move. Let $\operatorname{diam}(G)$ denote the diameter of the graph. The terminology "shortest path" will always refer to the graph distance (as opposed to say the sum of the edge-lengths). Aigner and Fromme in [1] proved the following useful result.

**Lemma 2.6 ([1])** *Let $G$ be any graph, $u, v \in V(G)$, $u \neq v$ and $P = \{u = v_0, v_1, \ldots v_s = v\}$ a shortest path between $u$ and $v$. A single cop $C$ can control $P$ after at most $\operatorname{diam}(G) + s$ moves.*

### 2.2.2 The cop's strategy

In the sequel, since we aim for a statement that holds a.a.s., we assume that we are given a realization of $\mathcal{G}(n, p, r)$ that is connected (which is true a.a.s. for our choice of parameters as, for instance, follows from the work of Penrose [15]) and for which the conclusions of Lemmas 2.2, 2.3 and 2.4 hold. We will show that under these conditions, a team of $21000\mathbb{L}$ cops is able to catch the robber. This will clearly prove the upper bound of Theorem 1.2.

Our strategy is an adaptation of the strategy of Aigner and Fromme showing $c(G) \leq 3$ for connected planar graphs. We will have three teams $T_1, T_2, T_3$ of cops, each consisting of $7000\mathbb{L}$ cops that are each charged with guarding a particular shortest path.

In more detail, a team $T_i$ that patrols a shortest path $P = v_0 v_1 \ldots v_m$ is divided into 7 subteams $T_{i,-3}, T_{i,-2}, T_{i,-1}, T_{i,0}, T_{i,1}, T_{i,2}, T_{i,3}$ of $1000\mathbb{L}$ cops each. These subteams will move in unison (i.e. the cops in a particular subteam will always be on the same vertex of $P$). The team $T_{i,0}$ moves exactly according to the strategy given by Lemma 2.6. That is, after an initial period, the $T_{i,0}$-cops are able to move along $P$ in such a way that, whenever the robber steps onto a vertex $v_k \in P$ then either the entire team $T_{i,0}$ is already on $v_k$ or they are on $v_{k-1}$ or $v_{k+1}$. Team $T_{i,j}$ will be $j$ places along $T_{i,0}$ (i.e. if $T_{i,0}$ is on $v_k$ then $T_{i,j}$ is on $v_{k+j}$). If this is not possible because $T_{i,0}$ is too close to the respective endpoint of $P$ then $T_{i,j}$ just stays on that endpoint (i.e. if $T_{i,0}$ is on $v_k$ and $k + j > m$ then $T_{i,j}$ is on $v_m$ and if $k + j < 0$

then $T_{i,j}$ stays on $v_0$). We now claim that the robber can not cross (in the sense that the edge she uses crosses an edge of $P$ when both are viewed as line segments) the path $P$ without getting caught by the cops of team $T_i$.

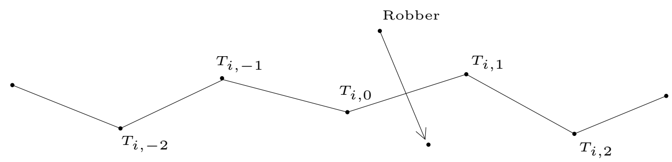


Figure 2: The robber tries to cross a path guarded by team $T_i$.

To see this, we first observe that if the robber moves along an edge that crosses some edge of $P$, then either her position before the move or her position right after the move is within distance at most $r/\sqrt{2}$ of some vertex of $P$ by Lemma 2.5. Next, we remark that whenever the robber stepw onto a vertex $u$ within distance $0.99 \cdot r$ of some vertex $v_k \in P$, then the cops can catch her in at most two further moves. This is because from $u$, the robber could move to $v_k$ in at most two moves (Lemma 2.4). As the cops of subteam $T_{i,0}$ follow the strategy prescribed by Lemma 2.6, they are guaranteed to be on one of $v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}$ when the robber arrives on $u$. But then there must be some team $T_{i,j}$ that inhabits the vertex $v_k$ at the very moment when the robber arrived on $u$. This team now acts as follows: at the time the robbers arrives on $u$, the subteam occupies the set $A$ provided by Lemma 2.2 (this one time the subteam do not all stay on the same vertex; instead they "spread" following the strategy implied by the lemma) and in the next move the cops are able to catch the robber, since they now dominate the closed neighbourhood of the vertex she inhabits. Thus, each of our three teams can indeed prevent the robber from crossing a chosen path (after an initiation phase). What is more, the robber can never get to within distance $0.99r$ of any vertex of such a path.

We can now mimic the strategy that Aigner and Fromme [1] developed for catching the robber on connected planar graphs using three cops. The idea is to confine the robber in smaller and smaller subgraphs of our graph, until finally the cops apprehend her. We start by taking two vertices $u, v$. We let $P_1$ be the shortest $uv$-path, and we let $P_2$ be the shortest $uv$-path in the graph with all internal vertices of $P_1$, and all edges that cross $P_1$ removed. (Using Lemmas 2.3 and 2.4 it is easily seen that at least one such path exists.) Note that $P_1 \cup P_2$ constitutes a Jordan curve and hence $\mathbb{R}^2 \setminus (P_1 \cup P_2)$ consists of two connected regions, the interior and the exterior. Once the game starts, we send $T_1$ to patrol $P_1$ and $T_2$ to patrol $P_2$. After an initial phase, the robber will either be trapped in the interior region or the exterior region of $\mathbb{R}^2 \setminus (P_1 \cup P_2)$. Let us denote the region she is trapped on by $R$. If it happens that every vertex inside $R$ is within distance $0.99r$ of some vertex of $P_1 \cup P_2$ then we are done by the previous argument. Let us thus assume this is not the case. We then proceed as follows: we remove all vertices not on $P_1 \cup P_2$ or inside $R$, and we remove all edges that cross $P_1$ or $P_2$. (Conceivably there can be vertices that lie inside $R$ but with an edge between them that passes through $P_1 \cup P_2$.) We let $P_3$ be a $uv$-path in the remaining graph that is shortest among all $uv$-paths that are distinct from $P_1, P_2$. (To see that at least one such path exists, we first find a vertex $u \in R$ that has distance at least $0.99r$ to every vertex of $P_1 \cup P_2$. Then we use Lemma 2.3 and 2.4 to construct vertex-disjoint paths between $u$ and two distinct vertices of $P_1 \cup P_2$.) See Figure 3 for a depiction.

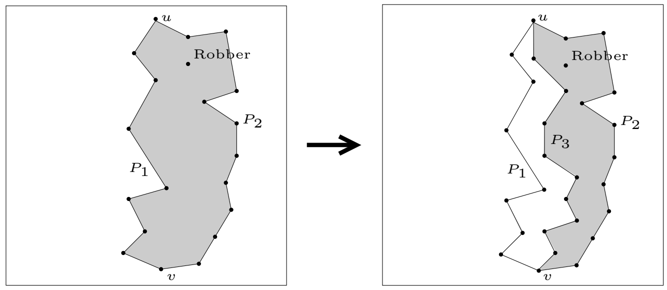


Figure 3: The adapted Aigner-Fromme strategy.

Note that $P_3$ does not cross $P_1$ or $P_2$ (but it may share some edges with them). In particular, $R \setminus P_3$ consists of two or more connected parts, each of which is either bounded by (parts of) $P_1$ and $P_3$ or by (parts of) $P_2$ and $P_3$. We now send $T_3$ to patrol $P_3$. After an initial phase, the robber will be caught in one of the connected parts $R'$ of $R \setminus P_3$. Without loss of generality $R'$ is bounded by $P_2, P_3$. Discarding unneeded parts of $P_2, P_3$ (namely those that do not bound $R'$) and relabelling we can also assume that $P_2, P_3$ only meet in their endpoints $u, v$. If every vertex inside $R'$ is within distance $0.99r$ of a vertex of $P_2, P_3$ we are again done. Otherwise, the team $T_1$ abandons guarding path $P_1$, we remove all vertices not on $P_2, P_3$ or inside $R'$ and all edges that cross $P_2$ or $P_3$, we find a $uv$-path $P_4$ in the remaining graph, shortest among all $uv$-paths different from $P_2, P_3$, and we let $T_1$ patrol $P_4$. Now $P_4$ will dissect $R'$ into two or more connected paths, and we repeat the procedure to either catch the robber or restrict her to an even smaller region.

It is clear that in each iteration of this process, at least one edge is removed from the subgraph under consideration. Hence the process must stop eventually. In other words, the robber will get caught eventually. This concludes the proof of (the upper bound of) Theorem 1.2.

### 2.3 Concluding remarks

As mentioned earlier, we conjecture that the $\Theta(\log n/p)$ in Theorem 1.2 can in fact be improved to $(1 + o(1)) \log_{1/(1-p)} n$.

We suspect that the $p^2$ term in the conditions for Theorem 1.2 is just an artefact of the proof and that the result should in fact hold when $pr^2 \geq n^{-1+\varepsilon}, p \leq 1 - \varepsilon$.

Let us also remark that bounding $p$ away from one is essential for our result, as can be seen for instance from Theorem 1.1 or the result in [3] that connected geometric graphs have bounded cop number. An interesting avenue of further investigation would thus be to see what goes on when $p \to 1$ as $n \to \infty$. Clearly some sort of phase change must occur, depending on the speed at which $p$ approaches one.

We remark that the proof of the lower bound in Theorem 1.2 readily generalizes to arbitrary dimensions (replacing $r^2$ by $r^d$ everywhere), but that the reasoning using in the upper bound proof is essentially two-dimensional. We would be very interested to learn of a proof technique that does work for all dimensions.

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