GRAPHS WITH AVERAGE DEGREE SMALLER THAN $\frac{30}{11}$ BURN SLOWLY

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Abstract. In this paper, we consider the following *firefighter problem* on a finite graph $G = (V, E)$. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit, a firefighter protects one vertex which is not yet on fire, and then the fire spreads to all unprotected neighbours of the vertices on fire. The objective of the firefighter is to save as many vertices as possible.

The surviving rate $\rho(G)$ of G is defined as the expected percentage of vertices that can be saved when a fire breaks out at a random vertex of G. Let $\varepsilon > 0$. We show that any graph G on n vertices with at most $(\frac{15}{11} - \varepsilon)n$ edges can be well protected, that is, $\rho(G) > \frac{\varepsilon}{60} > 0$. Moreover, a construction of a random graph is proposed to show that the constant $\frac{15}{11}$ cannot be improved.

1. Introduction

The following *firefighter problem* on a finite graph $G = (V, E)$ was introduced by Hartnell at the conference in 1995 [8]. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit, a firefighter protects one vertex which is not yet on fire, and then fire spreads to all unprotected neighbours of the vertices on fire. (Once a vertex is on fire or gets protected it stays in such state forever.) Since the graph is finite, at some point each vertex is either on fire or is protected by the firefighter, and the process is finished. (Alternatively, one can stop the process when no neighbour of the vertices on fire is unprotected. The fire will no longer spread. In such a case, we will still call all non-burning vertices protected, even though some of them were not selected by the firefigher.) The objective of the firefighter is to save as many vertices as possible. Today, 15 years later, our knowledge about this problem is much greater and a number of papers have been published. We would like to refer the reader to the survey of Finbow and MacGillivray for more information [13].

In this paper, we would like to focus on the following property. Let $\mathrm{sn}(G, v)$ denote the number of vertices in G the firefighter can save when a fire breaks out at vertex $v \in V$, assuming the best strategy is used. The surviving rate $\rho(G)$ of G, introduced in [11], is defined as the expected percentage of vertices that can be saved when a fire breaks out at a random vertex of G (uniform distribution is used), that is,

$$\rho(G) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v).$$

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It is not difficult to see that for cliques $\rho(K_n) = \frac{1}{n}$, since no matter where a fire breaks out only one vertex can be saved. For paths we get that

$$\rho(P_n) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G,v) = \frac{1}{n^2} \left(2(n-1) + (n-2)(n-2) \right) = 1 - \frac{2}{n} + \frac{2}{n^2}$$

(one can save all but one vertex when a fire breaks out at one of the leaves; otherwise two vertices are burned). It is not surprising that almost all vertices on a path can be saved, and in fact, all trees have this property. Cai, Cheng, Verbin, and Zhou [7] proved that the greedy strategy of Hartnell and Li [9] for trees saves at least $1 - \Theta(\log n/n)$ percentage of vertices on average for an n-vertex tree. Moreover, they managed to prove that for every outerplanar graph G, $\rho(G) \ge 1 - \Theta(\log n/n)$. Both results are asymptotically tight and improved earlier results of Cai and Wang [7]. However, this does not mean that it is easy to find the exact value of $\rho(G)$. It is known that the firefighter problem is NP-complete even for trees of maximum degree three [12].

The main focus of this paper is on sparse graphs. It is clear that sparse graphs are easier to control so their surviving rates should be relatively large. Finbow, Wang, and Wang [14] showed that any graph G with average degree strictly smaller than $8/3$ has the surviving rate bounded away from zero. Formally, it has been shown that any graph G with $n \ge 2$ vertices and $m \le (\frac{4}{3} - \varepsilon)n$ edges satisfies $\rho(G) \ge \frac{6\varepsilon}{5} > 0$, where $0 < \varepsilon < \frac{5}{24}$ is a fixed number. We improve their result by proving that any graph G with average degree strictly smaller than $30/11$ has the surviving rate bounded away from zero.

Theorem 1. *Suppose that graph G has $n \ge 2$ vertices and $m \le (\frac{15}{11} - \varepsilon)n$ edges, for some $0 < \varepsilon < \frac{1}{2}$. Then, $\rho(G) \ge \frac{\varepsilon}{60}$.*

(Let us note that our goal is to show that for sparse enough graphs the surviving rate is bounded away from zero, not to show the best lower bound for $\rho(G)$. To achieve this, any explicit bound, say, $\exp(-\varepsilon^{-10^{10}}) > 0$ would be as valuable as ours, and hence it is desired to aim for as simple argument as possible. The constant $\frac{1}{60}$ can be improved with more careful approach—we will discuss this briefly after the proof of Lemma 8.) Section 3 is devoted to proving this result.

On the other hand there are some dense graphs with large surviving rates (take, for example, a large collection of cliques). However, in Section 2 we construct a sparse random graph on n vertices with the surviving rate tending to zero as n goes to infinity. Hence the result is tight and the constant $\frac{15}{11}$ cannot be improved.

2. Random graphs are burning fast

The result of Finbow, Wang, and Wang [14] mentioned earlier is very interesting but it seems that with more complicated argument one should be able to improve their result, including slightly sparser graphs. But the question becomes how far this can be pushed? Where is the limit? Are there any sparse graphs that are burning fast? In other words, is there a family of sparse graphs in which the surviving rate is as close to zero as possible? The most natural graph to try is a random 3-regular graph and, indeed, a large random graph is burning fast with probability close to one. This implies that the constant $\frac{4}{3}$ in the result from [14] cannot be replaced by $\frac{3}{2}$. In order to warm up, we start with proving this fact (see Theorem 2).

Our results in this section refer to the probability space of random d-regular graphs with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, and asymptotics (such as 'asymptotically almost surely', which we abbreviate to a.a.s.) are for $n \to \infty$ with $d \geq 2$ fixed, and n even if d is odd.

Instead of working directly in the uniform probability space of random regular graphs on n vertices $\mathcal{G}_{n,d}$, we use the *pairing model* of random regular graphs, first introduced by Bollobás [4], which is described next. Suppose that dn is even, as in the case of random regular graphs, and consider dn points partitioned into n labeled buckets $v_1, v_2, \ldots, v_n$ of d points each. A *pairing* of these points is a perfect matching into $dn/2$ pairs. Given a pairing P, we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the buckets $v_1, v_2, \ldots, v_n$, and a pair $\{x, y\}$ in P corresponds to an edge $v_i v_j$ in $G(P)$ if x and y are contained in the buckets v_i and v_j, respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{(1-d^2)/4}$ depending on d but not on n. Therefore, any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$. For this reason, asymptotic results over random pairings suffice for our purposes. One of the advantages of using this model is that the pairs may be chosen sequentially so that the next pair is chosen uniformly at random over the remaining (unchosen) points. For more information on this model, see [17].

Theorem 2. *Let $G \in \mathcal{G}_{n,3}$. Then, a.a.s. $\rho(G) = O(\log n/n) = o(1)$.*

Note that Theorem 2 can be easily generalized to random d-regular graphs for any $d \geq 3$. Here, of course, we want to have as sparse graph as possible so we focus on $d = 3$. Let us also note that the result is tight, since (trivially) the diameter of any 3-regular graph is $\Omega(\log n)$, the process takes at least $\Omega(\log n)$ steps to finish and so at least $\Omega(\log n)$ vertices can be saved, regardless of the choice for the initial vertex at which a fire breaks out. In other words, Theorem 2 implies that a.a.s. the surviving rate of a random 3-regular graph is of order $\log n/n$. Before we move to the proof of this theorem, let us present a few important lemmas.

Lemma 3. *Let $K \geq 3$, $d \geq 3$ be fixed integers and $G \in \mathcal{G}_{n,d}$. The number of vertices that belong to a cycle of length at most K is at most $\log\log n$ a.a.s.*

This lemma is well known (see, for example, [5]). Since the proof is short and simple, we present it here for completeness.

Proof. Let $u \in V(G)$ and let $N_i(u)$ denote the set of vertices at distance at most i from u. Let f_i denote the number of vertices in a balanced d-regular tree, that is,

$$f_i = 1 + d\sum_{j=0}^{i-1}(d-1)^j = 1 + \frac{d((d-1)^i - 1)}{d-2} = O((d-1)^i).$$

It is clear that (deterministically) $n_i = |N_i(u)| \leq f_i$.

We will show that in the early stages of this process, the graphs grown from u's tend to be trees a.a.s. In other words, if we expose, step by step, the vertices at distance $1, 2, \ldots, i$ from u, we have to avoid, at step j, edges that induce cycles. So, we wish not to find edges between any two vertices at distance j from u or edges that join any two vertices at distance j to a common neighbour at distance $j+1$ from u. We shall refer to edges of this form as 'bad edges'. Note that the expected number of 'bad edges' at step $i+1$ is $O(n_i^2/n) = O(f_i^2/n) = O((d-1)^{2i}/n)$. Therefore, the expected number of 'bad edges' found up to step $i_1 = \lceil K/2 \rceil$ is equal to

$$\sum_{j=0}^{i_1-1} O\big((d-1)^{2j}/n\big) = O\big((d-1)^{2i_1}/n\big) = O\big(1/n\big)\,.$$

Thus, the expected number of vertices that belong to a cycle of length at most K is $O(1)$ and the assertion follows from Markov's inequality. $\square$

In order to show Theorem 2, we need to investigate the expansion properties of random d-regular graphs that follow from their eigenvalues. The adjacency matrix $A = A(G)$ of a given d-regular graph G with n vertices is an $n \times n$ real and symmetric matrix. Thus, the matrix A has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is known that certain properties of a d-regular graph are reflected in its spectrum but, since we focus on expansion properties, we are particularly interested in the following quantity: $\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|)$. In words, λ is the largest absolute value of an eigenvalue other than $\lambda_1 = d$ (for more details, see the general survey [10] about expanders, or [3], Chapter 9).

The value of λ for random d-regular graphs has been studied extensively. A major result due to Friedman [15] is the following:

Lemma 4 ([15])**.** *For every fixed $\varepsilon > 0$ and for $G \in \mathcal{G}_{n,d}$, a.a.s.*

$$\lambda(G) \leq 2\sqrt{d-1} + \varepsilon.$$

The number of edges $|E(S,T)|$ between two sets S and T in a random d-regular graph on n vertices is expected to be close to $d|S||T|/n$. A small λ (or large spectral gap) implies that this deviation is small. The following useful bound is essentially proved in [1] (see also [3]):

Lemma 5 (Expander Mixing Lemma)**.** *Let G be a d-regular graph with n vertices and set $\lambda = \lambda(G)$. Then for all $S, T \subseteq V$*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}\,.$$

(Note that $S \cap T$ does not have to be empty; in general, $|E(S,T)|$ is defined to be the number of edges between $S \setminus T$ to T plus twice the number of edges that contain only vertices of $S \cap T$.)

For our purpose here it is better to apply a slightly stronger lower estimate for $|E(S, V \setminus S)|$, namely,

$$|E(S, V \setminus S)| \geq \frac{(d-\lambda)|S||V \setminus S|}{n} \tag{1}$$

for all $S \subseteq V$. This is proved in [2], see also [3].

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Consider the vertex set U consisting of vertices that do not belong to a cycle of length at most 30. Since $|U| \geq n - \log\log n$ a.a.s. (by Lemma 3 applied with $K = 30$), it is enough to show that a.a.s. $\mathrm{sn}(G,u) = O(\log n)$ for all $u \in U$. Indeed, it this is shown, then a.a.s.

$$\begin{aligned}
\rho(G) &= \frac{1}{n^2}\sum_{v \in V} \mathrm{sn}(G,v) \\
&= \frac{1}{n^2}\sum_{v \in U} \mathrm{sn}(G,v) + \frac{1}{n^2}\sum_{v \in V \setminus U} \mathrm{sn}(G,v) \\
&= \frac{n - O(\log\log n)}{n^2} \times O(\log n) + \frac{O(\log\log n)}{n^2} \times O(n) \\
&= O(\log n / n).
\end{aligned}$$

Let $u \in U$ and let s_t denote the number of vertices burning at the end of time-step t. Since u does not belong to a short cycle, it is clear that in order to minimize s_t during a few first steps of the process, $t \in \{1, 2, \ldots, 15\}$, the firefighter should use a greedy strategy, that is, protect a vertex adjacent to the vertex on fire. If the greedy strategy is used, then at the end of time-step $t \in \{1, 2, \ldots, 15\}$ there are at least ρ_t vertices that catch fire in this time-step, where ρ_t's satisfy the following recursion: $\rho_1 = 2$ (during the first round, the firefighter protects one vertex but 2 other vertices catch fire) and $\rho_t = 2\rho_{t-1} - 1$ for $t \geq 2$ (there are $2\rho_{t-1}$ vertices adjacent to the fire, one of them gets protected). We get $\rho_t = 2^{t-1} + 1$ and, in particular, $s_{15} \geq \rho_{15} \geq 16000$.

Note that from (1) and Lemma 4 it follows that we may assume that

$$|E(S, V \setminus S)| \geq \frac{(d-\lambda)|S||V \setminus S|}{n} \geq 0.08|S|,$$

for all sets S of cardinality at most $n/2$. Therefore, if vertices from set S ($|S| \leq n/2$) are on fire, then at least $0.02|S|$ vertices are *not* on fire (including perhaps protected vertices) but are adjacent to at least one vertex on fire. Thus, at least

$$0.02 s_{15} - 16 \geq 0.01 s_{15} + 160 - 16 \geq 0.01 s_{15}$$

new vertices are going to catch fire at the next round and so $s_{16} \geq 1.01 s_{15}$. One can use this argument (inductively) to show that $s_{t+1} \geq 1.01 s_t$, provided that $s_t \leq n/2$.

Indeed, for any $t \geq 15$ with $s_t \leq n/2$, we have

$$s_{t+1} \geq 1.02 s_t - (t+1) \geq 1.01 s_t + 0.01 \cdot 16000 \cdot 1.01^{t-15} - (t+1) \geq 1.01 s_t.$$

This implies that at least $n/2$ vertices are on fire at time $t_0 \leq \log_{1.01} n$.

From this point on (that is, for $t > t_0$), it is easier to focus on $r_t = n - s_t$, the number of vertices that are *not* burning at time t. Using (1) and Lemma 4 again, we get that at least $0.02 r_t$ non-burning vertices are adjacent to some vertex on fire at time t. Thus, we get that $r_{t+1} = 0.99 r_t$ provided that $r_t \geq 100(t+1)$. Indeed, it follows that

$$r_{t+1} \leq 0.98 r_t + (t+1) \leq 0.99 r_t.$$

This implies that at least $n - O(\log n)$ vertices are on fire at time $T \leq \log_{1.01} n + \log_{1/0.99} n = O(\log n)$, and the proof is finished. $\square$

Now, consider a graph G with the maximum degree at most 3 and average degree at most $3 - \varepsilon$, for some $\varepsilon > 0$. This means that a positive fraction of all vertices (in fact, at least $\varepsilon n/3$ vertices) must have degree at most two. If the fire breaks out at such a vertex (which happens with probability at least $\varepsilon/3$), then one can use a greedy algorithm to protect an arbitrary vertex adjacent to the fire at any step of the process. By doing this, at most one new vertex catches fire at any time-step of the process and so one can save this way at least $1/3$ fraction of the vertices. (Note that the constant $1/3$ is best possible, as $\rho(K_3) = 1/3$. Moreover, this is not to say that the best protecting schedule is easy to determine—see [12] for more on that. Here, we just use a suboptimal strategy of protecting *any* vertex adjacent to the fire, which is enough for our purpose.) It follows that $\rho(G) \geq \varepsilon/9 > 0$. Therefore, in order to construct a sparser graph that burns fast, perhaps surprisingly, we need to introduce vertices of higher degree.

We propose the following random graph which proves that the constant $\frac{4}{3}$ cannot be replaced by $\frac{15}{11}$. Since the proof of this fact is similar to the one we used to show the result for random 3-regular graphs, we omit details, giving only a sketch of the proof. Fix $K \in \mathbb{N}$ that is congruent to 3 modulo 5, and $n \in \mathbb{N}$. Let us start with n disjoint paths of length K on the vertex set X, and the set Y consisting of $2n(K+2)/5$ isolated vertices (recall that $K+2$ is divisible by 5). Put random edges between X and Y such that every vertex in X has degree 4, and every vertex in Y has degree 5. It may happen that multiple edges are created but we restrict our probability space to simple graphs. It can be shown that we generate a simple graph with probability asymptotic to some constant $c > 0$, so the pairing model can be used again to study an asymptotic behaviour of this random graph. No edge between vertices in Y is added, so Y forms an independent set. Finally, subdivide each edge (both in the random part as well as in the n paths we started with) to get a graph $G(K,n)$. The random graph $G(K,n)$ is defined for any value of K (congruent to 3 modulo 5) but for our purpose we take, say, $K = (1+o(1)) \log\log\log n$ so that it is tending to infinity together with n but not so fast.

The properties of $G(K,n)$ are investigated next. Let us start with the average degree. We have $(1+o(1))Kn$ vertices of degree 4 and $(\frac{2}{5}+o(1))Kn$ vertices of degree 5. Thus, the number of edges in the graph G before subdividing edges is

$$\frac{1}{2}\sum_{v\in V}\deg_G(v) = (1+o(1))\frac{4\cdot Kn + 5\cdot \frac{2}{5}Kn}{2} = (1+o(1))3Kn,$$

so this is also the number of vertices of degree 2 after subdividing. Finally, the average degree of $G(K,n)$ can be estimated as follows:

$$\begin{aligned}\mathrm{Av}(G(K,n)) &= \frac{\sum_{v\in V}\deg_{G(K,n)}(v)}{|V(G)|} \\ &= (1+o(1))\frac{4\cdot Kn + 5\cdot\frac{2}{5}Kn + 2\cdot 3Kn}{Kn+\frac{2}{5}Kn+3Kn} = (1+o(1))\frac{30}{11}.\end{aligned}$$

(In fact, we could introduce $o(Kn)$ additional isolated vertices to make the average degree at most $\frac{30}{11}$ and this, of course, would not affect an asymptotic behaviour of the surviving rate of $G(K,n)$.)

Now, let us investigate how fast this graph is burning. One can show the following result.

Theorem 6. *Let* $K = (1+o(1))\log\log\log n$ *and* $K \equiv 3 \pmod 5$. *Then, a.a.s.*

$$\rho(G(K,n)) = o(1).$$

As we already mentioned, the argument that can be used to prove this theorem is similar to the one we used to prove this property for random 3-regular graphs. Unfortunately, we cannot use the spectral argument but the pairing model may be used to show the following property that we need. Suppose that some vertices of degree 5 are adjacent to two vertices on the same path (before edges are subdivided). If this happens, we call the path *bad*; otherwise the path is *good*. One can show that a.a.s. $o(n)$ paths are bad; in other words, a.a.s. almost all paths are good. Since our goal is to show the result that holds a.a.s., we may assume that the graph has this property.

Now, suppose that the fire breaks out on a vertex $v \in V = V(G(K,n))$. The firefighter must try to avoid hitting vertices of degree 5; otherwise the same argument as before can be used to show that the fire will start spreading exponentially fast and there is no hope to stop this process. It follows that $\mathrm{sn}(G,v) = o(Kn)$ if $\deg(v) = 5$. If v is on (or adjacent to) a bad path, then there is a chance to save a large fraction of the graph. However, since almost all paths are good, this does not affect an asymptotic behaviour of the surviving rate. Suppose then that v is on (or adjacent to) a good path. If $\deg(v) = 4$, the firefighter must start with protecting two vertices corresponding to the random part, but he has to give up after the next two rounds. The fire reaches a vertex of degree 5, and $\mathrm{sn}(G,v) = o(Kn)$ again. (See Figure 1.)

If $\deg(v) = 2$, the firefighter can keep pushing fire along the path (see Figure 2) but, since at the end of the path there is a vertex adjacent to 3 random edges, it is impossible to avoid vertices of degree 5. Once the fire reaches vertex of degree 5, it spreads quickly and the result holds.

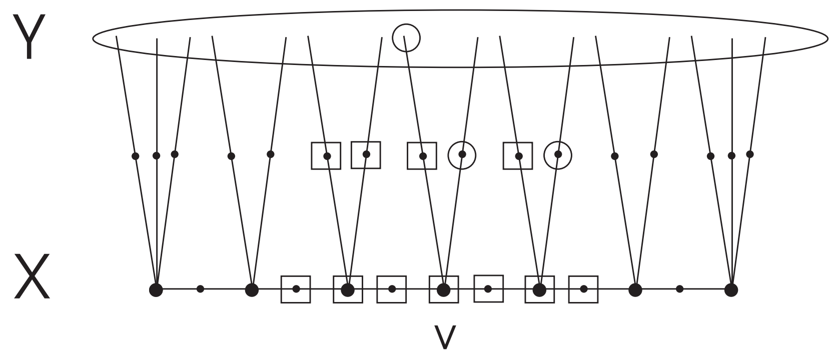


FIGURE 1. The fire starts on a degree 4 vertex (circles - protected vertices, squares - fire).

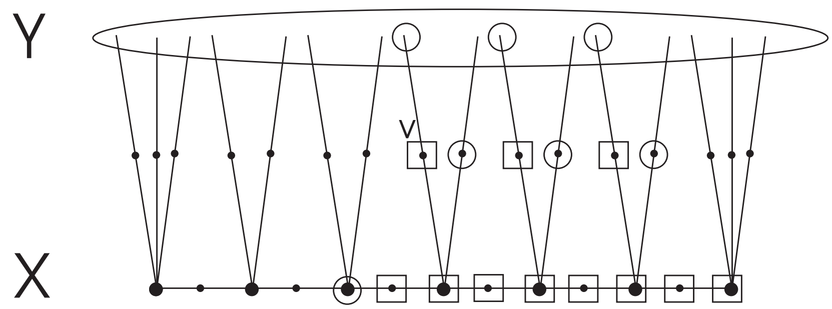


FIGURE 2. The fire starts on a degree 2 vertex (circles - protected vertices, squares - fire).

3. Sparse graphs can be well protected

In this section we show that all graphs with average degree smaller than $\frac{30}{11}$ have surviving rates bounded away from zero. In order to avoid some technical difficulties, we would like to focus on a special class of graphs $\mathcal{A}$: $G \in \mathcal{A}$ if and only if the minimum degree of G is at least two and no edge has both endpoints of degree two. The following lemma implies that a sparse graph G has a bounded surviving rate or there exists another sparse graph $G' \in \mathcal{A}$ with a small surviving rate, comparable to the one of G.

Lemma 7. *Suppose that graph G has $n \geq 2$ vertices and $m \leq (\frac{15}{11} - \varepsilon)n$ edges, for some $0 < \varepsilon < \frac{1}{2}$. Then,*

(a) $\rho(G) \geq \frac{\varepsilon}{40}$, *or*

(b) *there exists a graph $G' \in \mathcal{A}$ on N vertices, $M \leq (\frac{15}{11} - \frac{\varepsilon}{2})N$ edges, and $\rho(G') \leq \frac{\rho(G)}{0.9}$.*

Proof. We can assume that there are at most $\frac{\varepsilon}{20}n$ vertices of degree at most one; otherwise $\rho(G) \geq \frac{\varepsilon}{20} \cdot \frac{1}{2} = \frac{\varepsilon}{40}$ (with probability at least $\frac{\varepsilon}{20}$ the fire starts at such a vertex, and at least half of the vertices can be protected). Similarly, we can assume that at most $\frac{\varepsilon}{20}n$ edges have both endpoints of degree two; otherwise again $\rho(G) \geq \frac{\varepsilon}{40}$ (if the fire breaks out on such an endpoint, then we can protect one neighbour forcing the fire to move along the edge; we finish the process in the next round saving at least half of the vertices, unless $G = K_3$ for which the result trivially holds.).

Now, form G' from G by removing all vertices of degree at most one and contracting each edge with both endpoints of degree two into a single vertex. The number of vertices in G', N, satisfies $N \geq (1 - \frac{\varepsilon}{10})n$. The number of edges in G', M, can be bounded as follows

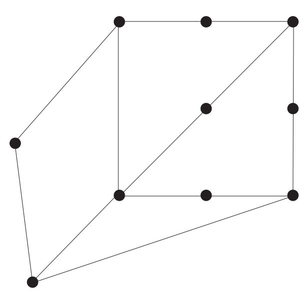
$$M \leq m \leq \left(\frac{15}{11} - \varepsilon\right) n \leq \frac{\frac{15}{11} - \varepsilon}{1 - \frac{\varepsilon}{10}} N \leq \left(\frac{15}{11} - \varepsilon\right)\left(1 + \frac{\varepsilon}{5}\right) N \leq \left(\frac{15}{11} - \frac{\varepsilon}{2}\right) N. \quad (2)$$

Finally,

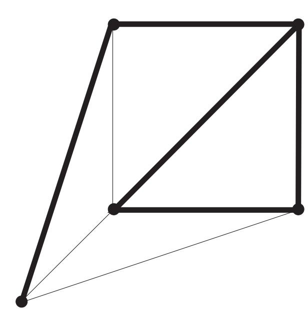
$$\rho(G') = \frac{1}{N^2} \sum_{v \in V(G')} \mathrm{sn}(G', v) \leq \frac{1}{(1 - \frac{\varepsilon}{10})^2 n^2} \sum_{v \in V(G')} \mathrm{sn}(G, v) \leq \frac{\rho(G)}{0.9}, \quad (3)$$

since $\varepsilon < \frac{1}{2}$. □

It follows from Lemma 7 that, without loss of generality, we can focus on graphs from the class $\mathcal{A}$. The next lemma has the same purpose (that is, to restrict our attention to even smaller subclass $\mathcal{B}$) and has a very similar proof. Before we define the class we would like to work with, let us introduce the following definition. The *kernel*, $\mathcal{K}[G]$, of a graph $G \in \mathcal{A}$ is the graph obtained from G by replacing each path (a, b, c) where $\deg(b) = 2$ by what we call a *long* edge. (Note that, since G is from $\mathcal{A}$, there is no edge with two endpoints of degree two and the kernel is well defined.) Note that the minimum degree of $\mathcal{K}[G]$ is 3. We refer to all other edges of $\mathcal{K}[G]$ as *short*. Finally, we remove all multiple edges, if necessary. See an example presented on Figure 3.



(a) graph G



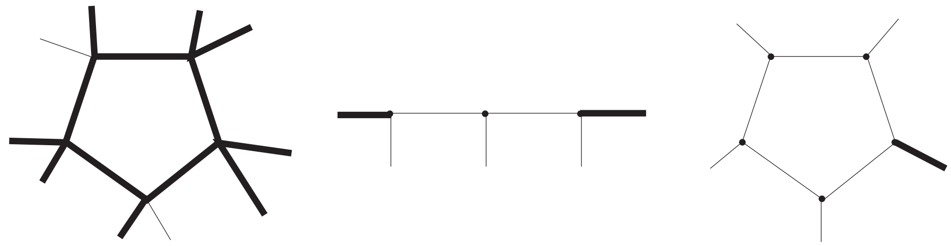
(b) kernel graph $\mathcal{K}[G]$ (*long* edges are thick)

FIGURE 3. Construction of kernels.

Now, we are ready to define the class $\mathcal{B}$: $G \in \mathcal{B}$ if and only if $G \in \mathcal{A}$ and $\mathcal{K}[G]$ has the following properties:

- (P1) there is no cycle consisting of long edges only, where each vertex has degree precisely 4 and is incident to at least 3 long edges,
- (P2) there is no path such that all internal vertices have degree 3, and both the first and the last edge is long,
- (P3) there is no cycle consisting of vertices of degree 3, where at least one vertex is adjacent to a long edge (not necessarily on the cycle).

(See Figure 4 for all forbidden configurations.)



(P1) (P2) (P3)

FIGURE 4. Forbidden configurations for class B.

The next lemma implies that a sparse graph $G \in \mathcal{A}$ has bounded surviving rate or there exists another sparse graph $G' \in \mathcal{B}$.

Lemma 8. *Suppose that graph $G \in \mathcal{A}$ has $n \geq 2$ vertices and $m \leq (\frac{15}{11} - \frac{\varepsilon}{2})n$ edges, for some $0 < \varepsilon < \frac{1}{2}$. Then,*

(a) *$\rho(G) \geq \frac{\varepsilon}{50}$, or*
(b) *there exists a graph $G' \in \mathcal{B}$ on N vertices and $M \leq (\frac{15}{11} - \frac{\varepsilon}{4})N$ edges.*

Proof. The proof is very similar to that of Lemma 7. There are at most $\frac{\varepsilon}{20}n$ long edges in $\mathcal{K}[G]$ associated with forbidden configurations (P1)-(P3); otherwise $\rho(G) \geq \frac{\varepsilon}{20} \cdot \frac{2}{5} = \frac{\varepsilon}{50}$ (note that if the fire breaks out on a long edge in the configuration (P1), then we are guaranteed to save only $\frac{2}{5}$ of the graph).

This time, we form G' from G by replacing all long edges in forbidden configurations by short ones (that is, contracting each path (a, b, c) in G with $\deg(b) = 2$ by a single edge). The number of vertices N in G' satisfies $N \geq (1 - \frac{\varepsilon}{20})n$. The number of edges M in G' can be bounded as follows

$$M \leq m \leq \left(\frac{15}{11} - \frac{\varepsilon}{2}\right) n \leq \frac{\frac{15}{11} - \frac{\varepsilon}{2}}{1 - \frac{\varepsilon}{20}} N \leq \left(\frac{15}{11} - \frac{\varepsilon}{2}\right)\left(1 + \frac{\varepsilon}{10}\right) N \leq \left(\frac{15}{11} - \frac{\varepsilon}{4}\right) N. \qquad (4)$$

The assertion follows. $\square$

Combining Lemma 7 and Lemma 8 we get that a sparse graph G either has the surviving rate bounded away from zero or there exists another sparse graph $G' \in \mathcal{B}$ with average degree smaller than $\frac{30}{11}$. We will show that the latter cannot happen (see Theorem 9), which proves that

$$\rho(G) \geq \min\left\{\frac{\varepsilon}{40}, 0.9\rho(G')\right\} \geq 0.9\frac{\varepsilon}{50} > \frac{\varepsilon}{60},$$

and the main result, Theorem 1, holds.

As we already mentioned, the constant $1/60$ can be improved slightly. Here, we mention briefly how one can do it, but still there are a lot of points in the existing argument that could be improved so one can push it further with even more careful approach. If $\rho(G) \geq c\varepsilon$ for some $c > 0$, then inequality (2) can be improved as follows:

$$\frac{M}{N} \leq \frac{\frac{15}{11} - \varepsilon}{1 - 2c\varepsilon} = \frac{15}{11} - \varepsilon \left(1 - \frac{30}{11}c\right) + O(\varepsilon^2),$$

and (3) can be replaced by $\rho(G') = (1 + O(\varepsilon^2))\rho(G)$. With this in hand, (4) can be done more carefully to get

$$\frac{M}{N} \leq \frac{\frac{11}{15} - \varepsilon\left(1 - \frac{30}{11}c\right) + O(\varepsilon^2)}{1 - \frac{5}{2}c\varepsilon + O(\varepsilon^2)} = \frac{15}{11} - \varepsilon \left(1 - \frac{135}{22}c\right) + O(\varepsilon^2).$$

This implies that the constant $1/60 \approx 0.0167$ could be easily replaced by a constant that is arbitrarily close (but smaller than) $\frac{22}{135} \approx 0.163$ (almost 10 times larger).

Theorem 9. *Every graph $G \in \mathcal{B}$ on $n \geq 2$ vertices has $m > \frac{15}{11}n$ edges.*

Proof. Take any graph $G = (V, E) \in \mathcal{B}$, $V = \{v_1, v_2, \ldots, v_n\}$. We consider the kernel $\mathcal{K}[G]$ and use the model similar to the *pairing model*, which was used in the previous section. (However, unlike in the case of random regular graphs there is *no* randomness involved in this case, but rather some degeneracy/discharging-type argument.) Consider $2m$ points partitioned into n labeled buckets $v_1, v_2, \ldots, v_n$; bucket with label v_i consists of $\deg(v_i)$ points. A *pairing* of these points is a perfect matching into m pairs. Given a graph G, we may construct a pairing $P(G)$, as follows: the vertices are the buckets $v_1, v_2, \ldots, v_n$, and a pair $\{x, y\}$ in P corresponds to an edge v_iv_j in $G(P)$ if x and y are contained in the buckets v_i and v_j, respectively.

We construct the pairing $P(G)$ introducing edges of G one by one (however, sometimes it would be more convenient to introduce a few edges at the time). At time $t = 0$ we have no pair of points matched. At every time-step t, we would like to control the following values:

$$\begin{aligned} N_t &= \text{the number of saturated points (that is, already paired);} \\ D_t &= \sum_{i \in [n]} \frac{\text{the number of saturated points in the bucket with label } v_i}{\deg(v_i)}. \end{aligned}$$

For a given t, let $n = N_t - N_{t-1}$ and $d = D_t - D_{t-1}$. The *weight* of the operation performed at time t is n/d. Clearly, at the beginning of the process $D_0 = N_0 = 0$, whereas at the end of the process, time T, we have $N_T = 2|E|$, $D_T = |V|$; the average degree in G is then N_T/D_T.

Let us partition the vertex set V into the following subsets:

(i) V_3 : vertices of degree 3;
(ii) $V_4^{\geq 3}$: vertices of degree 4 that are adjacent to at least 3 long edges;
(iii) $V_4^{\leq 2}$: vertices of degree 4 that are adjacent to at most 2 long edges;
(iv) $V_{\geq 5}$: vertices of degree at least 5.

Since only a small number of long edges can be adjacent to vertices from V_3 (see properties (P2) and (P3)), in order to minimize the average degree, a large number of vertices of degree 4 adjacent to long edges would have to be introduced. However, it is also not possible (see the property (P1)) and so no graph from the family $\mathcal{B}$ is sparse. We will make this observation rigorous soon.

In order to warm up and to explain the strategy that will be used, we assume first that $V_3 = V_4^{\le 2} = \emptyset$ and that all edges are long. Let $x = |V_4^{\ge 3}|$. The assertion clearly holds if $x = 0$. The notation below ([13] and [12]) might be misleading (at this point) but we prepare the setting for a future argument.

[13] We start with adding x_{13} long edges between vertices in $V_4^{\ge 3}$. It follows from (P1) that $x_{13} \le x - 1$ (the set $V_4^{\ge 3}$ cannot induce a long cycle). Adding one edge to P increases N_t by $n_{13} = 4$ (4 points since an edge is long), and D_t by $d_{13} = \frac{2}{2} + \frac{2}{4} = \frac{3}{2}$ (two points in a bucket of degree two, and two points in a bucket of degree 4). The weight of each operation is $\frac{8}{3} \approx 2.667$.

[12] Add x_{12} edges with at least one endpoint in $V_{\ge 5}$; this time, $n_{12} = 4$ and $d_{12} \le \frac{2}{2} + \frac{1}{4} + \frac{1}{5} = \frac{29}{20}$ (two points in bucket of degree two, one point in a bucket of degree at least 4, and one point in a bucket of degree at least 5). The weight is at least $\frac{80}{29} \approx 2.759$.

Thus, the average degree in G can be bounded as follows

$$\mathrm{Av}(G) \ge \frac{x_{13}n_{13} + x_{12}n_{12}}{x_{13}d_{13} + x_{12}d_{12}} = \frac{4x_{13} + 4x_{12}}{\frac{3}{2}x_{13} + \frac{29}{20}x_{12}}.$$

As we already mentioned, $x_{13} \le x-1$, so there are a few edges between vertices in $V_4^{\ge 3}$. This implies that there are a lot of edges between $V_4^{\ge 3}$ and $V_{\ge 5}$. In fact, by a simple counting argument we get that $2x_{13} + x_{12} \ge 4x$. It is not difficult to show that $\mathrm{Av}(G)$ is maximized for $x_{13} = x-1$ and $x_{12} = 4x - 2x_{13}$, that is, when $V_4^{\ge 3}$ induces a tree and all other edges are between $V_4^{\ge 3}$ and vertices of degree 5. (Note that the random graph we constructed in Section 2 to show that the result is sharp has exactly this form.) We get,

$$\mathrm{Av}(G) \ge \frac{4(x-1) + 4(2x+2)}{3(x-1)/2 + 29(2x+2)/20} = \frac{12x+4}{22x/5 + 7/5} > 30/11,$$

and the assertion holds.

In general we have many more other operations to consider, but the idea is exactly the same. We start with adding a long edge ab ($\deg(a) \ge 4$, $\deg(b) = 3$) together with edges bc and bd that are adjacent to b (note that those edges are short by property (P2)). If there are other long edges attached to c or d that are adjacent to other vertices of degree at least 4, then we add them as well (note that if this is the case, then $\deg(c) \ge 4$ and $\deg(d) \ge 4$ by (P2)). There are four cases to consider that are illustrated in Figure 5. We perform x_i operations of type $[i]$.

[1] Add 3 long and two short edges in one round; $n_1 \ge 16$, $d_1 \le \frac{6}{2} + \frac{3}{3} + \frac{7}{4}$; the weight is at least $\frac{64}{23} > \frac{80}{29}$.

[2] Add two long and two short edges; $n_2 \ge 12$, $d_2 \le \frac{4}{2} + \frac{4}{3} + \frac{4}{4}$; the weight is at least $\frac{36}{13} > \frac{80}{29}$.

[3] Add one long edge and at least one of c, d has degree at least 4; $n_3 \ge 8$, $d_3 \le \frac{2}{2} + \frac{4}{3} + \frac{2}{4}$; the weight is at least $\frac{48}{17} > \frac{80}{29}$.

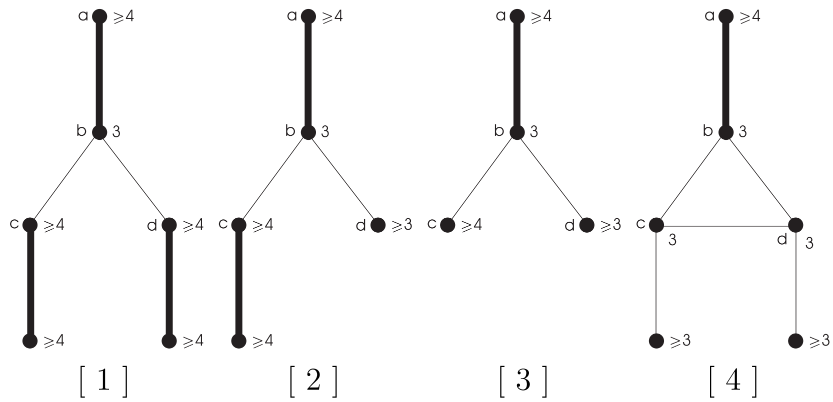


FIGURE 5. First 4 types of rounds.

[4] Add one long edge and both c and d has degree 3. In this situation, we add all edges adjacent to c and d as well (note that those edges are short by (P2)); $n_4 \geq 14$, $d_3 \leq \frac{2}{2} + \frac{11}{3} + \frac{1}{4}$, the weight is at least $\frac{168}{59} > \frac{80}{29}$ (the extreme configuration is presented in Figure 5).

In the next couple of rounds we add a long edge ab (this time $\deg(a) = \deg(b) = 3$) together with all (short by (P2)) edges adjacent to a and b. (Note that we must have 4 distinct short edges by (P3).) As before, we add one long edge (if such an edge exists) incident to each neighbour of a and b. Such an edge must have both endpoints of degree at least 4 by (P2) and the fact that we already introduced all long edges between vertices of degree 3 and 4. All five, almost identical, cases are illustrated in Figure 6.

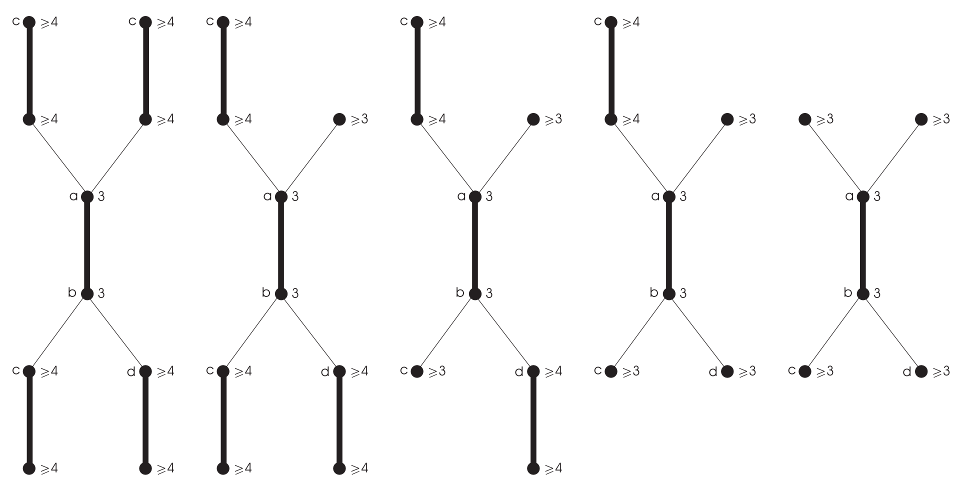


FIGURE 6. Next 5 types of rounds: [5]-[9].

[5] Add 5 long and 4 short edges; $n_5 \geq 28$, $d_5 \leq \frac{10}{2} + \frac{6}{3} + \frac{12}{4}$; the weight is at least $\frac{14}{5} > \frac{80}{29}$.

[6] Add 4 long and 4 short edges; $n_6 \geq 24$, $d_6 \leq \frac{8}{2} + \frac{7}{3} + \frac{9}{4}$, the weight is at least $\frac{288}{103} > \frac{80}{29}$.

[7] Add 3 long and 4 short edges; $n_7 \geq 20$, $d_7 \leq \frac{6}{2} + \frac{8}{3} + \frac{6}{4}$, the weight is at least $\frac{120}{43} > \frac{80}{29}$.

[8] Add two long and 4 short edges; $n_8 \geq 16$, $d_8 \leq \frac{4}{2} + \frac{9}{3} + \frac{3}{4}$, the weight is at least $\frac{64}{23} > \frac{80}{29}$.

[9] Add one long and 4 short edges; $n_9 \geq 12$, $d_9 \leq \frac{2}{2} + \frac{10}{3}$, the weight is at least $\frac{36}{13} > \frac{80}{29}$.

Now, we add a long edge ab ($a \in V_4^{\leq 2}$). Since all long edges incident to vertices of degree 3 are already introduced, $\deg(b) \geq 4$. Since $a \in V_4^{\leq 2}$ is incident to at least two short edges, there must be a short edge ac, which was not yet introduced (that is why we kept adding additional long edges if it was possible). We add ac together with an additional long edge cd if such an edge exists (note that in such a case, both vertices must have degree at least 4).

[10] Add two long and one short edge; $n_{10} \geq 10$, $d_{10} \leq \frac{4}{2} + \frac{6}{4}$; the weight is at least $\frac{20}{7} > \frac{80}{29}$.

[11] Add one long and one short edge; $n_{11} \geq 6$, $d_{11} \leq \frac{2}{2} + \frac{1}{3} + \frac{3}{4}$; the weight is at least $\frac{72}{25} > \frac{80}{29}$.

Finally,

[12] Add long edges with at least one endpoint is in V_5; the weight is at least $\frac{80}{29}$ (we discussed this before).

[13] Add long edges between vertices in $V_4^{\geq 3}$; the weight is $\frac{8}{3}$ (again, we discussed this before).

[14] Add all short edges; $n_{14} = 2$, $d_{14} = 2/3$, the weight is 3.

The average degree in G is

$$\mathrm{Av}(G) \geq \frac{\sum_{i=1}^{14} x_i n_i}{\sum_{i=1}^{14} x_i d_i}.$$

The smallest possible weight of $\frac{8}{3}$ is the one corresponding to operation [13] but, as we discussed before, we cannot have many operations of this type by (P1). In fact, we get that $x_{13} \leq x - 1$, where $x = |V_4^{\geq 3}|$. Next in the line is the weight of $\frac{80}{29}$ corresponding to [12]. Since

$$3x_1 + 2x_2 + x_3 + x_4 + 4x_5 + 3x_6 + 2x_7 + x_8 + 2x_{10} + x_{11} + x_{12} + 2x_{13} \geq 4x,$$

it is not hard to see that the previous construction minimizes the average degree and the assertion follows. $\square$

4. More firefighters

Let $k \in \mathbb{N}$ and suppose that in each subsequent time unit, firefighters protect k vertices which are not yet on fire. For this natural extension (the $k = 1$ case corresponds to the original problem studied in this paper), one can define the surviving rate $\rho_k(G)$ analogously ($\rho_1(G) = \rho(G)$ and clearly $\rho_\ell(G) \geq \rho_k(G)$ whenever $\ell > k$).

It turns out (perhaps surprisingly) that $k \geq 2$ is much easier to analyze comparing to the $k = 1$ case. Let $\tau_k = k + 2 - \frac{1}{k+2}$. In [16], it has been shown recently that for any $\varepsilon > 0$ and $k \geq 2$, every graph G on n vertices with at most $(\tau_k - \varepsilon)n$ edges is not flammable; that is, $\rho_k(G) > \frac{2\varepsilon}{5\tau_k} > 0$. Moreover, a construction of a family of flammable random graphs is proposed to show that the constant τ_k cannot be improved.

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