Elimination schemes and Lattices

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Abstract

Perfect vertex elimination schemes are part of the characterizations for several classes of graphs, including chordal and cop-win. Partial elimination schemes reduce a graph to an important subgraph, for example, k-cores and robber-win graphs. We are interested in those partial elimination schemes, in which once a vertex can be eliminated it is always ready to be eliminated. In such a scheme, the sets of subsets of eliminated vertices, when ordered by inclusion, form an upper locally distributed lattice. We also consider the cop-win orderings having this property, the lattices obtained from the process of cleaning graphs, and raise the following question: which graphs are associated with distributive lattices?

Key words: searching, upper locally distributive lattice, chip-firing, Pse-ordering, brush number, cop number, k-cores, simplicial elimination ordering, domination elimination ordering, chordal.

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1 Introduction

Elimination schemes occur in a variety of contexts in graph theory. Consider the following examples.

- A *chordal* graph is characterized by being able to successively remove a vertex x whose closed neighbourhood $N[x]$ is a complete graph. Such a vertex x is called *simplicial* and the ordering is also called simplicial. See [4] for many references to this problem.
- The k-*core* of a graph (or hypergraph) is the largest subgraph of minimum degree at least k. The k-core can be found by the *vertex deletion algorithm* that repeatedly deletes vertices with degree less then k. This algorithm always terminates with the k-core of the graph, which is possibly empty; see [20] for example.

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- In *corner elimination*, a vertex x can be eliminated if there is a vertex y such that $N[x] \subseteq N[y]$. A vertex x can be eliminated by *domination* if its open neighbourhood, $N(x)$ is contained in the open neighbourhood of another vertex y, i.e., $N(x) \subseteq N(y)$. A graph is cop-win [17] if it can be reduced to a single vertex by eliminating corners; it is tandem-win [6] if it can be reduced by a domination ordering. In both cases, if the graph is not eliminated then this is the subgraph in which the robber looks for a winning strategy.

- In cleaning a graph [16], a vertex v can be eliminated if it has at least as many brushes as incident dirty edges; when eliminated, it passes a brush from v along each incident dirty edge that becomes clean. The question is: what is the fewest number of brushes required?

In [4], all the orderings are *perfect*, that is, all the vertices are eliminated. Our results cover perfect as well as partial elimination schemes. Informally, we consider partial elimination schemes in which once a vertex is ready to be eliminated it stays in that state regardless of which other vertices are eliminated. Formally, fix a graph G, with n vertices, and a property P. Let $A = \{v_1, v_2, \ldots, v_m\} \subseteq V(G)$, we set G_0^A to be G and, for $i \geq 1$, G_i^A to be the induced subgraph on $V(G) - \{v_1, v_2, \ldots, v_i\}$. A vertex of G will be called *primed* if it satisfies property P. The set $A = \{v_1, v_2, \ldots, v_m\}$ is a *P-elimination scheme* if v_i is primed in G_{i-1}^A for $1 \leq i \leq m$; it is a *P-strong-elimination ordering*, Pse-ordering for short, if whenever $x \neq v_i$ and x is primed in G_{i-1}^A then it is primed in G_i^A again for $1 \leq i \leq m$. Note that A need not include all the vertices of G. Finally, G is a *P-strong-elimination graph* if every P-elimination scheme is also a P-strong-elimination ordering.

Given a graph G, property P, and a Pse-order I, there is an associated subset C_I of eliminated vertices which we will refer to as a *configuration*. Configuration C_I is *reachable* from configuration C_J if the Pse-order J can be extended to I; in symbols, $J \to I$. Given G and P, let $Con(G, P) = \{C_I \ : I \text{ is a Pse-ordering}\}$. Note that two different orderings I and J could have $C_I = C_J$.

Recall that a *lattice* is a partial order in which every pair of elements have a least upper bound and a greatest lower bound. A finite lattice in which every element has a unique representation as the meet of meet-irreducibles (defined in Section 2) is called *upper locally distributive* (ULD) lattice. Caspard showed the following result.

Theorem 1.1. *[5] A lattice is upper locally distributive if and only if the interval between an element and the join of its upper covers forms a boolean lattice (i.e. a complemented distributive lattice).*

In the next section, our main result, Theorem 2.5, shows that for a Pse-ordering P, the set $Con(G, P)$ ordered by inclusion forms an upper-locally distributive lattice. In Section 3, we show that for all graphs both the processes of constructing a k-core and for cleaning a graph give Pse-orderings. In Section 3.1, we give sufficient conditions on a graph so that the cop-win dismantling procedure is a Pse-ordering. In each context, we raise the question: *which graphs are associated with distributive lattices?* and *which upper-locally distributive lattices are associated with the context?*

Upper distributive lattices also occur in chip-firing [7, 10, 12, 13, 14, 18]. Indeed, the similarities between cleaning a graph and chip-firing was the original motivation to this work.

2 Pse-orderings form an Upper Locally Distributive Lattice

Let I and J be Pse-orders both on subsets of $V(G)$. Define (I, J) to be the order I followed by the elements of $J - I$ preserving the order in J.

Lemma 2.1. *Given graph G and property P, let I and J be Pse-orders. The ordering (I, J) is a Pse-ordering.*

Proof. Consider the orderings $I = \langle a_1, a_2, \ldots, a_i \rangle$ and $J = \langle b_1, b_2, \ldots, b_j \rangle$. Let $(I, J) = \langle a_1, a_2, \ldots, a_i, c_1, c_2, \ldots, c_k \rangle$. Clearly, a_r can be eliminated once $a_1, a_2, \ldots, a_{r-1}$ have been eliminated. Consider $c_r = b_s \in J - I$ and suppose that $a_1, a_2, \ldots, a_i, c_1, c_2, \ldots, c_{r-1}$ have been eliminated. Since $\{b_1, b_2, \ldots, b_{s-1}\} \subseteq \{a_1, a_2, \ldots, a_i, c_1, c_2, \ldots, c_{r-1}\}$, it follows that $b_1, b_2, \ldots, b_{s-1}$ have been eliminated, which is a sufficient condition for $b_s = c_r$ to be ready to be eliminated. □

Lemma 2.2. *Given G and property P, let I, J be Pse-orderings. Then $I \to J$ if and only if $C_I \subseteq C_J$.*

Proof. That $I \to J$ implies $C_I \subseteq C_J$ follows immediately from the definition of the process.

Suppose $C_I \subseteq C_J$. By Lemma 2.1, (I, J) is also a Pse-ordering that eliminates C_J. Moreover, it follows immediately from the definition of the Pse-ordering (I, J) that when (I, J) is used, configuration C_I is reached first before the process arrives at configuration C_J. As a result, $I \to J$. □

We now show that the set $Con(G, P)$ with $\subseteq$ together form an upper locally distributive lattice. First, we need to show that it forms a partially ordered set. Let $\mathcal{G}_P = (Con(G, P), \subseteq)$.

Theorem 2.3. *Given G and property P, the relation $\mathcal{G}_P$ is a partially ordered set.*

Proof. Since $I \to J$ is equivalent to $C_I \subseteq C_J$, $\mathcal{G}_P$ is equivalent to a set of subsets of $V(G)$ ordered by inclusion and hence is a partially ordered set. □

There is a problem when dealing with a subset that may have come from any number of orderings: going in the reverse direction intuitively might make sense, but it is not usually well defined. To avoid that in our next definition, we set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and whenever there is a choice of primed vertices, we automatically choose the vertex with lower index. The *common configuration*, $C_{\langle I,J \rangle}$, of I and J is obtained by the following procedure: let $C_0 \subseteq C_I \cap C_J$ be all the vertices with property P in G and $\overline{C_0}$ be the sequence formed by taking the vertices in the order of their indices; and recursively $C_j \subseteq (C_I \cap C_J) - \bigcup_{i=0}^{j-1} C_i$ be all the vertices with property P in $G - \bigcup_{i=0}^{j-1} C_i$. Form the sequence $\overline{C_j}$ by adding the vertices of C_j in order of their indices to $\overline{C_{j-1}}$. Continue until there are no more primed vertices. This produces a Pse-ordering (again biased by the original ordering of $V(G)$) which we will denote by $\langle I, J \rangle$.

Given G and some property P, take $C_I, C_J \in Con(G, P)$. Let $C_I \vee C_J$ be the configuration $C_I \cup C_J$. Let $C_I \wedge C_J = C_{\langle I,J \rangle}$ be the configuration obtained from $\langle I, J \rangle$. In the next result, we will show that $A \vee B$ and $A \wedge B$ are in fact the least upper and the greatest lower bounds of subsets A and B, often called the *join* and the *meet* respectively, proving that $\mathcal{G}_P$ is a lattice. Note that the meet does not correspond to the intersection of sets. For example, in the path $P_3 = (a, b, c)$ where property P is 'a vertex is simplicial', $\{a, b\}$ and $\{b, c\}$ are configurations but $\{b\}$ is not.

Theorem 2.4. *Given G and some property P, the partially ordered set $\mathcal{G}_P$ is a lattice.*

Proof. Take $C_I, C_J \in \mathcal{G}_P$. As both sequences I and J are Pse-orderings, by Lemma 2.1, $K = (I, J)$ is a Pse-ordering. Moreover, this Pse-ordering results in the set $C_K = C_I \cup C_J$, hence, C_K is an upper bound of C_I and C_J. It remains to show that it is the least one. If $C_I \subseteq C_{K'}$ and $C_J \subseteq C_{K'}$ then it follows that $C_K = C_I \cup C_J \subseteq C_{K'}$ so $K \to K'$ and hence C_K is the join of C_I and C_J in the partial order.

It is a basic fact of lattice theory that a finite partially ordered set with a bottom element and a join operation is a lattice. However, since it is of independent interest, we give the meet operation explicitly.

By construction, $C_{\langle I,J\rangle} \subseteq C_I \cap C_J$ so $C_{\langle I,J\rangle} \subseteq C_I$ and $C_{\langle I,J\rangle} \subseteq C_J$. Hence $C_{\langle I,J\rangle}$ is a lower bound of C_I and C_J, and it remains to show that it is the greatest one. If $C_K \neq C_{\langle I,J\rangle}$ is a lower bound of C_I and C_J then $C_{\langle I,J\rangle} \vee C_K$ is also a lower bound, so we may assume that $C_{\langle I,J\rangle} \subseteq C_K$. In eliminating the vertices of C_K, we may eliminate the vertices of $C_{\langle I,J\rangle}$ first. For a contradiction, suppose that x is the first vertex of $C_K - C_{\langle I,J\rangle}$ that is eliminated. Now $x \notin C_I \cap C_J$ else it would be in $\langle I, J\rangle$. Therefore either $x \in C_I - C_J$ or $x \in C_J - C_I$. In the first case, $C_{\langle I,J\rangle} \cup \{x\} \not\subseteq C_J$ and in the second one $C_{\langle I,J\rangle} \cup \{x\} \not\subseteq C_I$. This is a contradiction, since $C_{\langle I,J\rangle} \cup \{x\} \subseteq C_K$ and K is a lower bound of I and J. Since x does not exist, it follows that $\langle I, J\rangle = H$ and so $C_{\langle I,J\rangle}$ is the greatest lower bound of C_I and C_J. □

In general, we will refer to $\mathcal{G}_P$ as a *Pse-lattice*. Since we will now be referring to lattices and partially ordered sets we will use the more conventional $\leq$.

Let L be a lattice. Then $a \in L$ is an *upper cover* of $b \in L$ if $a < b$ and there is no element c where $a < c < b$; a is *meet-irreducible* if $a = b \wedge c$ implies $a = b$ or $a = c$; a is *join-irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$. Note that the meet-irreducibles of L correspond to the elements with only one upper cover; join-irreducibles to those having one lower cover. The *interval* $[a, b]$ in a lattice is $[a, b] = \{x : a \leq x \leq b\}$.

Chip-firing lattices are upper locally distributive (see [7, 10, 12, 13, 14, 18]) and we now prove that are Pse-lattices are also upper locally distributive (abbreviated ULD).

Theorem 2.5. *Given a graph G and a property P, then $\mathcal{G}_P$ is upper locally distributive.*

Proof. Let $A \in \mathcal{G}_P$ and set $U = \{u_1, u_2, \ldots, u_m\}$ be the set of primed vertices in A. Let $A_i = A \cup \{u_i\}$ for $i \in \{1, 2, \ldots, m\}$. These are the upper covers of A.

Let 1_A be the join of the upper covers of A and let $\mathcal{I}$ be the interval between $[A, 1_A]$. Since each upper cover of A is obtained by eliminating a single element of U, we get that $1_A = A \cup U$. Also recall that every element of U stays primed regardless of how many other elements of U have been eliminated. It follows, therefore, that for every subset I of $\{1, 2, \ldots, m\}$ we have $\vee_{i\in I} A_i \in \mathcal{I}$ and if $J, K \subseteq \{1, 2, \ldots, m\}$ and $J \neq K$ then $\vee_{j\in J} A_j \neq \vee_{k\in K} A_k$.

Suppose $M \in \mathcal{I}$. Then $M \subseteq 1_A$ and hence $M = A \cup U'$ where $U' \subseteq U$ and therefore $M = \vee_{u_i\in U'} A_i$. Suppose now that $M, N \in \mathcal{I}$ so that $M = \vee_{i\in W} A_i$ and $N = \vee_{i\in Y} A_i$, where $W, Y \subseteq \{1, 2, \ldots, m\}$. Certainly, $M \vee N = M \cup N$ and $A \subseteq M \cup N \subseteq 1_A$ thus $M \vee N \in \mathcal{I}$. Consider $M \wedge N$. Now $M \cap N = A \cup \{u_i : i \in W \cap Y\}$ and each vertex of $\{u_i : i \in W \cap Y\} \subseteq U$ is ready to be eliminated in configuration A. Hence, $M \wedge N = M \cap N$ and so $M \wedge N \in \mathcal{I}$. This proves that the interval $\mathcal{I}$ is isomorphic to the lattice of all subsets of U ordered by inclusion, i.e., a boolean lattice. From Theorem 1.1 it follows that $\mathcal{G}_P$ is an upper locally distributive lattice. □

3 Examples of Pse Orderings

In this section, we provide a few simple Pse-lattices and analyze two slightly more sophisticated ones. In Section 3.1 we consider the reduction to robber-win graphs in the usual cops-and-robber game as well as in the tandem-win version. In Section 3.2, we consider the lattices obtained from the process of cleaning a graph.

A vertex is called *simplicial* if its neighbourhood is a clique. *Chordal graphs* have a simplicial elimination scheme—a vertex is primed when its neighbourhood is a clique. In general, if the neighbourhood of x, denoted $N(x)$, in G is a clique then in $H = G - \{z\}$, $z \neq x$, $N(x) - \{z\}$ is also a clique. Thus, a simplicial elimination scheme is a Pse-order. Recall that we do not insist on an elimination scheme that eliminates all the vertices. In particular, if a graph G has no simplicial vertex then $\mathcal{G}_P$ consists of the empty set.

Theorem 3.1. *Let G be a graph and P the property that '$N(x)$ is a clique'. Then $\mathcal{G}_P$ is a ULD lattice.*

k-cores: For a given k, the property P is that $\deg(x) < k$. Once a vertex has degree less than k, eliminating other vertices can never increase its degree, therefore this is a strong-elimination property.

Theorem 3.2. *Let G be a graph, and property P is '$\deg(x) < k$'. Then $(Con(G, P), \leq)$ is a ULD lattice.*

An acyclic directed graph G can be regarded as a prerequisite structure, such as scheduling. Taking the transitive closure of G adds no extra restrictions on G and results in a partial order. Hence, we will assume that G is a poset. A *topological sort* of G is formed by taking some vertex, a, with no incoming edges—usually called a source of the graph or a minimal element in the poset—removing this from G and continuing in $G - \{a\}$. Note that once a vertex, x, is a minimal element, removing other vertices can never change this. Therefore, with the property P being *'vertex x is a minimal element,'* then a topological sort is a Pse-ordering. We will need Birkhoff's theorem which states:

Theorem 3.3. *(Birkhoff) [3] A lattice is distributive if and only if it is isomorphic to the lattice of the ideals of the order induced on its join-irreducibles.*

Theorem 3.4. *Let G be an acyclic directed graph and Property P is 'x is a minimal element' then $\mathcal{G}_P$ is distributive.*

Proof. In G, an *order ideal* is a set C of vertices with the property that if $x \in C$ and $y < x$ then $y \in C$. If $C \in \mathcal{G}_P$ then C is an order ideal and any order ideal is in $\mathcal{G}_P$. Thus $\mathcal{G}_P$ is the set of order ideals of G and it follows from Birkhoff's theorem that $\mathcal{G}_P$ is distributive. □

3.1 Copwin Graphs and Domination Elimination schemes

The game of Cops-and-Robber was introduced in [17, 21], see also [2]. The characterization of graphs that require just one cop, called *copwin*, is via an elimination scheme. Vertex $x \in V(G)$ is *primed* if either $V(G) = \{x\}$ or there is a vertex $y \neq x$ such that $N[x] \subseteq N[y]$. Informally, if x is primed, then in the context of Cops-and-Robber, it is called a *corner*: where the robber can be trapped by the cop being located at y (which we will call an *apex*). A graph G is cop-win if there is an elimination scheme that reduces G to a single vertex. We artificially add that a if G is a singleton then it can be eliminated; it is a 'degenerate' corner. This does not apply to an isolated vertex in a larger graph. Now, a graph is cop-win if and only if it has a perfect elimination scheme via corners.

In a *domination elimination scheme*, vertex x is primed if there is a $y \neq x$ such that $N(x) \subseteq N(y)$; we say $N(x)$ is dominated. The difference between the two schemes is that the domination elimination scheme does not require that x is adjacent to y.

If we again add that if G is a singleton then it can be eliminated then in [6], a sufficient, but not necessary, condition for a tandem pair of cops to win is the existence of a perfect domination elimination scheme.

Neither of these elimination schemes are Pse-orders. For example, consider graph D in Figure 1.

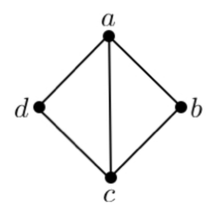

Figure 1: Graph D

Under both elimination schemes, all vertices are primed, but if a is removed then c is no longer primed. Note that the copwin scheme requires the edge ac, but C_4 is a counter example for the domination elimination scheme. Eliminating the graph in Figure 1 gives a sufficient condition.

Theorem 3.5. *Let G be a connected graph with no induced subgraph isomorphic to D and let P be the property of 'x is a corner'. The partially ordered set $(Con(G, P), \leq)$ is a ULD lattice.*

Proof. Let G be a connected graph with no induced subgraph isomorphic to D. We need only prove that any elimination ordering is a Pse-order. In order to do that, we prove the following two claims.

Claim 1: If x, y are corners in G, then x is a corner in $G - \{y\}$.

If there is an apex z, $z \not\sim y$, for x then z is still an apex for x in $G - \{y\}$. Therefore, we may assume that y is the only apex of x. However, if $z \neq x$ is an apex for y then it is also an apex for x. Hence x is the only apex for y and consequently $N(x) - \{y\} = N(y) - \{x\}$. Call this set N and note that $N = N(x) \cap N(y)$. If $N = \emptyset$ and since G is connected it follows that G is isomorphic to K_2 in which case x is a 'degenerate' corner in $G - \{y\}$. If N is a clique then any vertex of N is an apex for x contrary to our assumption. If $w, v \in V(N)$ and $w \not\sim v$ then the subgraph induced by $\{x, y, v, w\}$ is isomorphic to D, again a contradiction. Hence, x is a corner of $G - \{y\}$.

Claim 2: If x is a corner in G and G is connected and D-free, then $G - \{x\}$ is connected and D-free.

The property that $G - \{x\}$ is D-free clearly holds, since an edge is eliminated only when an endpoint is eliminated. Since G is connected, there is a path from x to every vertex of G. Let y be an apex for x in G. Since $N[x] \subseteq N[y]$, there is a path from y to every vertex of $G - \{x\}$ and so $G - \{x\}$ is connected. The proof of the claim is complete.

The theorem follows immediately now since once x is a corner it remains a corner until it is eliminated and thus for connected, D-free graphs, the corner elimination scheme is a Pse-order. □

In fact, the proof of Theorem 3.5 shows that the Pse-orders are a subset of simplicial orders. There is a similar theorem for domination elimination schemes. The proof is also similar to that of Theorem 3.5 so we leave it to the reader.

Theorem 3.6. *Let G be a connected graph with no induced subgraph isomorphic to D or to C_4, and let P the property that '$N(x)$ is dominated'. Then $(Con(G, P), \leq)$ is a ULD lattice.*

Note that in both cases, the elimination of the last vertex is necessary else a perfect corner elimination scheme would give rise to a partial order that is a UDL lattice minus $V(G)$, the top element. Also note that the assumption of connectedness is not necessary since otherwise the ULD lattice would be the direct product of the lattices of the components. Unfortunately, without connectedness we would have non-copwin graphs having a perfect elimination scheme which is not desirable.

3.2 Cleaning Lattices

Imagine a network of pipes that must be periodically cleaned of a contaminant that regenerates, say algae. In *cleaning* such a network (see [1, 8, 9, 11, 15, 16, 19] for example), there is an initial configuration of brushes on vertices; every vertex and edge is regarded as *dirty*. A vertex is 'ready to be cleaned' if it has at least as many brushes as incident dirty edges. These are our primed vertices. A primed vertex may *clean* whereupon it sends one brush along each incident dirty edge which is now said to be *clean*: no brush traverses a clean edge. The vertex is also deemed to be clean. Excess brushes remain on the clean vertex and take no further part in the process. Figure 3.2 illustrates the cleaning process for a graph G where there are initially 2 brushes at vertex a. The solid edges indicate dirty edges while the dotted edges indicate clean edges. For example, the process starts with vertex a being cleaned, sending a brush to each of vertices b, c. We note the number of brushes on the vertex and whether it has been cleaned by the subscript f.

$$(a2, b0, c0, d0), (a_f0, b1, c1, d0), (a_f0, b_f0, c2, d0), (a_f0, b_f0, c_f1, d1), (a_f0, b_f0, c_f1, d_f1).$$

For cleaning, a graph G comes with an initial distribution of brushes: every time a vertex cleans it and incident edges are eliminated from the graph. Property P is that 'x has at least as many brushes as incident dirty edges'. Clearly, once a vertex is primed, it remains primed until it is cleaned, therefore this is a strong-elimination property.

Theorem 3.7. *Let G be a graph with an initial distribution of brushes Ω on G and property P is that 'x has at least as many brushes as incident dirty edges'. Then $(Con(G, P), \leq)$ is a ULD lattice.*

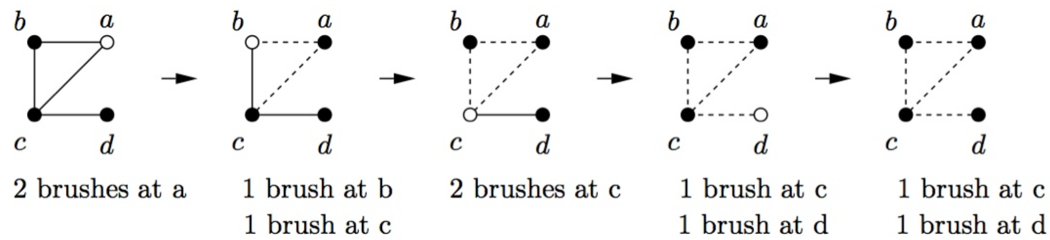

Figure 2: An example of the cleaning process for graph G.

4 Cleaning and Distributive Lattices

If a lattice is ULD and also 'downward-locally-distributive' then it would be *distributive*. Figure 3, the lattice obtained from cleaning the path $P_3 = (a, b, c)$, shows that this is not always true for cleaning lattices. However, in this section we show that for each distributive lattice L, there is a graph G and an initial configuration Ω so that L is isomorphic to $\mathcal{G}_P$, although it is not always the case that the configuration Ω will clean all vertices of G.

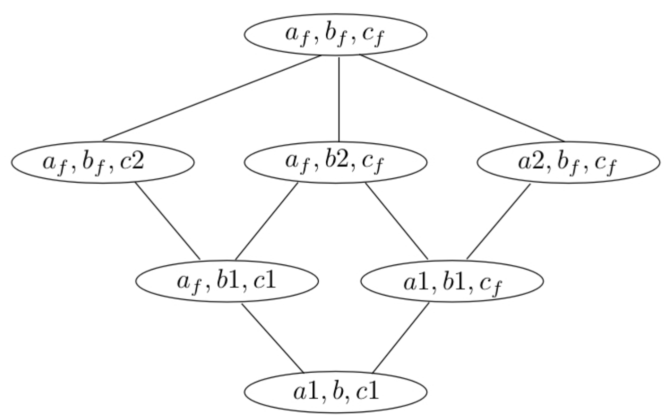

Figure 3: Configurations for P_3 beginning with a brush at each leaf.

In [16], there is a characterization of graphs and configurations with a unique Pse-ordering. Such graphs and configurations would give rise to a lattice that was a single chain (i.e. every pair of configurations are comparable).

Theorem 4.1. *Let L be a distributive lattice. Then there is a graph G and an initial configuration Ω so that L is isomorphic to $\mathcal{G}_P$.*

Proof. Let L be a distributive lattice and $M(L) = \{m_1, m_2, \ldots, m_k\}$ be the *set of meet-irreducibles* of L. The meet-irreducibles inherit the order from L. We create a graph G with $V(G) = \{m_1, m_2, \ldots, m_k\} \cup \{r_1, r_2, \ldots, r_k\}$ and $E(G)$ consists of the Hasse Diagram of $M(L)$ (on $\{m_1, m_2, \ldots, m_k\}$); $\{r_1, r_2, \ldots, r_k\}$ is a clique and m_i is adjacent to $r_1, r_2, \ldots, r_j$ where $j = \max\{0, d^+(m_i) - d^-(m_i)\}$ where $d^-(m_i)$ is the number of upper covers of m_i in $M(L)$ and $d^+(m_i)$ is the number of lower covers. Let J be the set of minimal elements in $M(L)$. The original configuration will be $w(x) = \max\{0, d^-(x) - d^+(x)\}$ if $x \in M(L)$ and 0 otherwise. Any Pse-ordering must be a topological sort of $M(L)$. That is, only the minimals of $M(L)$ can clean at the outset; after, by construction a vertex can clean if and only if all of its lower covers, and consequently all of the vertices below it in $M(L)$, have been cleaned. Thus a point in a Pse-ordering corresponds to an order ideal of $M(L)$. The Pse-orderings are ordered by set inclusion. No vertex in $\{r_1, r_2, \ldots, r_{k+1}\}$ can be cleaned since they have at most k brushes but require $k+1$ to clean.

To complete the proof, we invoke Birkhoff's theorem (Theorem 3.3). We started with a distributive lattice L, obtained a graph G and produced a distributive lattice L' isomorphic to the lattice of ideals of $M(L)$. Therefore L is isomorphic to L'. □

Not all distributive lattices are cleaning lattices of the cleaning of a whole graph, as shown in the next example. The sets of clean vertices are indicated in Figure 4. In the following examples, we use the notation $x \sim y$ to indicate that x and y are adjacent vertices.

Example 4.2. *For a contradiction, suppose that there exists a graph G and an initial configuration Ω that cleans the whole graph and yields the lattice presented in Figure 4. The graph G has 4 vertices. Vertices a and b are initially ready to be cleaned, but c, d are not. After cleaning vertex b but not after cleaning vertex a, vertex c is ready to be cleaned, hence $b \sim c$ and $a \not\sim c$. Vertex d is ready to be cleaned only after a and b have been cleaned, i.e., $a \sim d \sim b$ and now d has two brushes. Since d was not ready to be cleaned after only one of a and b are cleaned, it must be adjacent to c, i.e.,* $\deg(d) = 3$. *Thus after both b and c have been cleaned d will have two brushes and one dirty adjacent vertex (a) and thus should be able to be cleaned contradicting the fact that it can be cleaned only after a.*

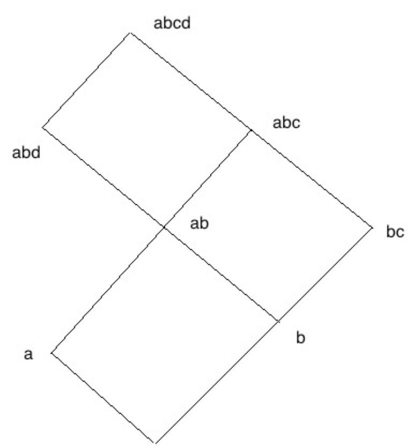

Figure 4: A distributive lattice that is not the cleaning lattice of a whole graph.

The next example shows that not every ULD lattice is a cleaning lattice of a subgraph of a simple graph.

Example 4.3. *Consider the lattice in Figure 5. Similarly to the previous example, d can be cleaned only after a and b, so $a \sim d \sim b$, or after b and c have been cleaned so $b \sim d \sim c$. In* *both cases, d has received two brushes and two incident edges have been cleaned and can now be cleaned. However, after a and c have been cleaned the same situation arises but d is shown as not able to be cleaned, a contradiction.*

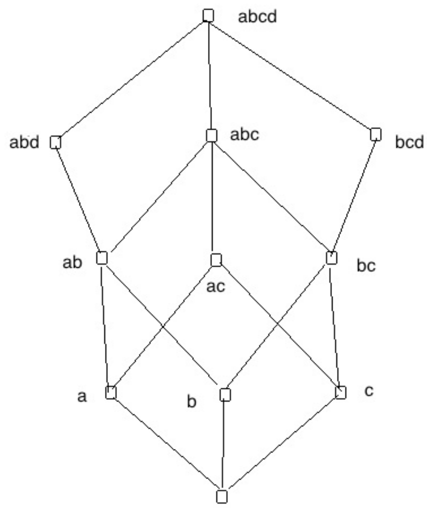

Figure 5: A UDL that is not the cleaning lattice.

This leaves us with several question concerning the other elimination schemes.

Question 4.4. *Which graphs under simplicial-elimination give rise to distributive lattices? And to each distributive lattice is there an associated graph?*

What happens if we restrict to perfect elimination schemes?

Question 4.5. *Which chordal graphs and cop-win graphs give rise to distributive lattices? And to each distributive lattice is there an associated chordal and cop-win graph?*

For chordal graphs, the subclass of interval graphs would be a candidate but the 'sun' graph gives a distributive lattice.

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