Modular orientations of random and quasi-random regular graphs

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Abstract

Extending an old conjecture of Tutte, Jaeger conjectured in 1988 that for any fixed integer $p \geq 1$, the edges of any $4p$-edge connected graph can be oriented so that the difference between the outdegree and the indegree of each vertex is divisible by $2p+1$. It is known that it suffices to prove this conjecture for $(4p+1)$-regular, $4p$-edge connected graphs. Here we show that there exists a finite p_0 so that for every $p > p_0$ the assertion of the conjecture holds for all $(4p+1)$-regular graphs that satisfy some mild quasi-random properties, namely, the absolute value of each of their nontrivial eigenvalues is at most $c_1 p^{2/3}$ and the neighborhood of each vertex contains at most $c_2 p^{3/2}$ edges, where $c_1, c_2 > 0$ are two absolute constants. In particular, this implies that for $p > p_0$ the assertion of the conjecture holds asymptotically almost surely for random $(4p+1)$-regular graphs.

1 Introduction

A nowhere-zero 3-flow in an undirected graph $G = (V, E)$ is an orientation of its edges and a function f assigning a number $f(e) \in \{1, 2\}$ to any oriented edge e such that for any vertex $v \in V$,

$$\sum_{e \in D^+(v)} f(e) - \sum_{e \in D^-(v)} f(e) = 0,$$

where $D^+(v)$ is the set of all edges emanating from v, and $D^-(v)$ is the set of all edges entering v.

A well known conjecture of Tutte, raised in 1966 in [19], asserts that any 4-edge connected graph admits a nowhere-zero 3-flow. This conjecture is still wide open, and it is not even known whether or not there is a finite k so that any k-edge connected graph has a nowhere-zero 3-flow, although it is known that if the edge connectivity of an n-vertex graph is at least $c \log_2 n$, then it does have a nowhere-zero 3-flow. This is proved in [4] (in a somewhat implicit, stronger form, with $c = 2$), and in [14] (with $c = 4$).

It is known (see, e.g., [17]) that a graph admits a nowhere-zero 3-flow if and only if it has a nowhere-zero flow over Z_3, or equivalently, an edge orientation in which the difference between the outdegree and the indegree of any vertex is divisible by 3. It is also known (see, e.g., [8]) that it is enough to prove the conjecture for 5-regular graphs. Thus, Tutte's conjecture has the following equivalent form.

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Research supported in part by an ERC Advanced grant, by a USA-Israeli BSF grant, by the Oswald Veblen Fund and by the Bell Companies Fellowship.
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Conjecture 1.1 (Tutte) *Every* 4*-edge connected* 5*-regular graph has an edge orientation in which every outdegree is either* 4 *or* 1.

Jaeger [12] extended this statement and conjectured that for any integer $p \geq 1$, the edges of any $4p$ edge-connected graph can be oriented so that the difference between the outdegree and the indegree of any vertex is divisible by $2p+1$. Such an orientation is called a mod $(2p+1)$-orientation. Similarly as before, it is known that the general case can be reduced to the $(4p+1)$-regular one, and thus the conjecture has the following equivalent form.

Conjecture 1.2 (Jaeger's modular orientation conjecture) *For any fixed integer* $p \geq 1$*, every* $4p$*-edge connected,* $(4p+1)$*-regular graph has a mod* $(2p+1)$*-orientation, that is, an edge orientation in which every outdegree is either* $3p+1$ *or* p.

This conjecture is still open, and appears to be difficult. It is thus natural to try and prove that its assertion holds for almost all $(4p+1)$-regular graphs. (It is known that a typical $(4p+1)$-regular graph is $(4p+1)$-edge connected.) Our main result in this note is that the assertion of the conjecture holds for all $(4p+1)$-regular graphs with a sufficiently large eigenvalue gap and with no dense neighborhoods, for all sufficiently large p. As a special case this implies that the assertion holds for almost all $(4p+1)$-regular graphs. In order to state the main result we need the notion of an (n, d, λ)-graph.

An (n, d, λ)-graph is a d-regular graph on n vertices in which the absolute value of any nontrivial eigenvalue of the adjacency matrix is at most λ. This notation was introduced by the first author, motivated by several results showing that if λ is significantly smaller than d then the graph exhibits some strong pseudo-random properties.

Theorem 1.3 *There are absolute positive constants* d_0, c_1, c_2 *so that if* $\lambda < c_1 d^{2/3}$*, then any* (n, d, λ)*-graph* $G = (V, E)$*, where* $d = 4p+1 > d_0$*, in which no neighborhood of a vertex contains more than* $c_2 d^{3/2}$ *edges, has a mod* $(2p+1)$*-orientation, that is, an orientation in which every outdegree is either* $3p+1$ *or* p.

In order to prove the main result it is convenient to consider an equivalent formulation of Conjecture 1.2, proved in [8] for $p = 1$ and in [13] for general p. The equivalence is a consequence of an old result of Hakimi [10] which follows from Hall's Theorem, or from the maxflow mincut Theorem. For two disjoint sets of vertices S and T in a graph $G = (V, E)$, let $E(S, T)$ denote the set of all edges with an end in S and an end in T, and let $S^c = V \setminus S$ denote the complement of S.

Theorem 1.4 ([13]) *Let* $p > 0$ *be an integer, and let* G *be a* $(4p+1)$*-regular graph. Then* $G = (V, E)$ *has an orientation in which every outdegree is either* $3p+1$ *or* p *if and only if there is a partition* $V = V^+ \cup V^-$ *with* $|V^+| = |V^-| = |V|/2$ *such that for all* $S \subseteq V$,

$$|E(S, S^c)| \geq (2p+1)\big||S \cap V^+| - |S \cap V^-|\big|. \tag{1}$$

In view of the above, the following result implies the assertion of Theorem 1.3.

Theorem 1.5 *There are absolute positive constants* d_0, c_1, c_2, c_3 *so that if* $d > d_0$ *and* $\lambda < c_1 d^{2/3}$*, then any* (n, d, λ)*-graph* $G = (V, E)$ *with an even number of vertices in which no neighborhood of a vertex contains more than* $c_2 d^{3/2}$ *edges, has a vertex partition* $V = V^+ \cup V^-$ *with* $|V^+| = |V^-| = |V|/2$ *such that for all* $S \subseteq V(G)$,

$$|E(S, S^c)| \geq \left(\frac{d}{2} + c_3\sqrt{d}\right)\big||S \cap V^+| - |S \cap V^-|\big|. \tag{2}$$

The above theorem implies, as a special case, that the assertion of Conjecture 1.2 holds for almost all $d = (4p + 1)$-regular graphs. This refers to the probability space of random $d = (4p + 1)$-regular graphs with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, where d is a fixed integer. We say that a property holds in this space 'asymptotically almost surely' (or a.a.s., for short) if the probability that a member $G \in \mathcal{G}_{n,d}$ satisfies the property tends to 1 as n tends to ∞ (n is even since d is odd). See, e.g., [7], [20] for more details about $\mathcal{G}_{n,d}$.

Theorem 1.6 *There exists a finite p_0 so that for any fixed integer $p > p_0$, a random $(4p + 1)$-regular graph G admits, a.a.s., a mod $(2p+1)$-orientation, that is, an orientation in which every outdegree is either $3p + 1$ or p.*

The rest of this note is organized as follows. In Section 2 we present a few useful lemmas. The main result, Theorem 1.5 (which implies Theorem 1.3), is proved in Section 3. Section 4 contains the simple derivation of Theorem 1.6 from the main result, and the final section contains some concluding remarks and open problems. Throughout the note we assume, whenever this is needed, that the number n of vertices of the graphs considered is sufficiently large as a function of their degree of regularity d.

2 Preliminaries

To prove the result we use the expansion properties of random d-regular graphs that follow from their eigenvalues. The adjacency matrix $A = A(G)$ of a given d-regular graph G on n vertices, is an $n \times n$ real symmetric matrix. Thus, the matrix A has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is known that several structural properties of a d-regular graph are reflected in its spectrum. Since we focus on expansion properties, we are particularly interested in the following quantity: $\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|)$. In words, λ is the largest absolute value of an eigenvalue other than $\lambda_1 = d$ (for more details, see the general survey [11] about expanders, or [6], Chapter 9).

The number of edges $|E(S, T)|$ between two sets S and T in a random d-regular graph on n vertices is expected to be close to $d|S||T|/n$. A small λ (that is, a large spectral gap) implies that the deviation is small. The following useful bound is essentially proved in [2] (see also [6]):

Lemma 2.1 (The Expander Mixing Lemma) *Let G be a d-regular graph with n vertices and set $\lambda = \lambda(G)$. Then for all $S, T \subseteq V$*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}\,.$$

When $T = S^c$ is the complement of S, it will be sometimes convenient to apply the following lower estimate for $|E(S, S^c)|$,

$$|E(S,S^c)| \geq \frac{(d-\lambda)|S||S^c|}{n} \tag{3}$$

for all $S \subseteq V$. This is proved in [5] (see also [6]).

We also need the well known fact (see [1], [15]) that for fixed d and large n, any (n, d, λ)-regular graph satisfies

$$\lambda \geq (2 - o(1))\sqrt{d-1}. \tag{4}$$

For a partition (A, A^c) of the vertex set, define

$$\delta(A, A^c) = |E(A, A^c)| - \frac{d|A||A^c|}{n},$$

that is, $\delta(A, A^c)$ measures the difference between the actual number of edges between A and A^c and the expected value of this number in a graph of edge density d/n. The following simple lemma shows that for a small λ, if two partitions are not too far from each other, then the sizes of the two corresponding cuts are similar.

Lemma 2.2 *Let G be a d-regular graph with n vertices and set $\lambda = \lambda(G)$. For any two partitions (A, A^c), (B, B^c) of the vertex set with*

$$|A \setminus B| + |B \setminus A| = x,$$

we have

$$|\delta(A, A^c) - \delta(B, B^c)| \leq 4\lambda\sqrt{xn}.$$

Proof: For any two partitions (A, A^c), (B, B^c),

$$\begin{aligned} |\delta(A, A^c) - \delta(B, B^c)| &\leq \left| E(A \cap B, A^c \cap B) - \frac{d|A \cap B||A^c \cap B|}{n} \right| \\ &+ \left| E(A \cap B, A \cap B^c) - \frac{d|A \cap B||A \cap B^c|}{n} \right| \\ &+ \left| E(A^c \cap B^c, A \cap B^c) - \frac{d|A^c \cap B^c||A \cap B^c|}{n} \right| \\ &+ \left| E(A^c \cap B^c, A^c \cap B) - \frac{d|A^c \cap B^c||A^c \cap B|}{n} \right| \\ &\leq 4\lambda\sqrt{xn}, \end{aligned}$$

where the last inequality follows from Lemma 2.1. ∎

3 The proof of the main result

In this section we prove Theorem 1.5, that is, we show that a d-regular graph $G = (V, E)$ with a large spectral gap and no dense neighborhoods, with $d \geq d_0$ for some positive integer d_0, has a partition (V^+, V^-) of V with $|V^+| = |V^-| = n/2$, where $n = |V|$ is even, such that the condition (2) holds for any $S \subseteq V$. Note that for $S = V^+$ (or $S = V^-$) this gives

$$|E(V^+, V^-)| \geq (\frac{d}{2} + c_3\sqrt{d})|V^+| = \frac{dn}{4} + \Omega(\sqrt{d}n).$$

Therefore, it is natural to start with a proof that there is such a dense bisection.

We need the following result proved in [3].

Lemma 3.1 ([3]) *There are two absolute constants $b_1, b_2 > 0$ such that the following holds. Any d-regular graph in which the neighborhood of any vertex contains at most $b_1 d^{3/2}$ edges, has a cut of size at least $\frac{dn}{4} + b_2 n\sqrt{d}$.*

Note that, in particular, the condition of the theorem holds for any graph in which no edge is contained in more than $b_1\sqrt{d}$ triangles. Using this lemma, we prove that in fact one can always ensure a large bisection, that is, a cut in which the two vertex classes are of equal size.

Theorem 3.2 *There are absolute constants $d_0, b_1, b_3 > 0$ so that the following holds. Let $G = (V, E)$ be a d-regular graph on an even number of vertices n, where $d \geq d_0$, in which the neighborhood of any vertex contains at most $b_1 d^{3/2}$ edges. Then V has a cut (V^+, V^-) such that $|V^+| = |V^-| = n/2$ and*

$$|E(V^+, V^-)| \geq \left(\frac{d}{4} + b_3\sqrt{d}\right) n.$$

Proof: By Lemma 3.1 there is a cut (A, B) of G of size $|E(A, B)| \geq \frac{nd}{4} + b_2 n\sqrt{d}$. Without loss of generality assume that $|A| \geq |B|$. Define $b_2' = \min\{\frac{b_2}{4}, \frac{1}{4}\}$ and $b_3 = \frac{b_2'}{2}$. If $|A| = |B|$, there is nothing to prove. Otherwise, we prove the existence of the required bisection by shifting vertices from A to B until they have equal sizes. For each vertex $v \in A$, let $d_C(v)$ denote the degree of the vertex v in the cut (A, B), that is, its number of neighbors in B.

Starting with the cut (A, B) consider, first, the case $|A| \geq (\frac{1}{2} + \frac{1}{\sqrt{d}})n$. In this case, if for every $v \in A$, $d_C(v) \geq \frac{d}{2}$, then after shifting any vertex from A to B the size of the new cut is still at least

$$(\frac{1}{2} + \frac{1}{\sqrt{d}})n\frac{d}{2} - d \geq \frac{dn}{4} + b_2'\sqrt{d}n.$$

Otherwise, there is a vertex $v \in A$ with $d_C(v) < \frac{d}{2}$, and we can shift it to B and increase the size of the cut. Keeping this process we obtain a cut (A, B) (with the modified sets A, B generated), which is of size at least $\frac{dn}{4} + b_2'\sqrt{d}n$, in which $|B| \leq |A| \leq (\frac{1}{2} + \frac{1}{\sqrt{d}})n$.

If, now, for any vertex $v \in A$, $d_C(v) \geq \frac{d}{2} + b_2\sqrt{d}$, then after shifting an arbitray vertex from A to B we obtain a new cut of size at least

$$\frac{n}{2}(\frac{d}{2} + b_2\sqrt{d}) - d > \frac{dn}{4} + b_2'\sqrt{d}n.$$

Else, we can shift a vertex v with $d_C(v) < \frac{d}{2} + b_2\sqrt{d}$ from A to B, decreasing the size of the cut by less than $2b_2\sqrt{d}$. As there are at most $\frac{n}{\sqrt{d}}$ required steps until A and B are of the same size, and in the end of each step either the size of the cut is above $\frac{dn}{4} + b_2'\sqrt{d}n$ or the size decreases by at most $2b_2\sqrt{d}$, we conclude that there is a bisection of size at least

$$\frac{dn}{4} + b_2'\sqrt{d}n - \frac{n}{\sqrt{d}}2b_2\sqrt{d} = \frac{dn}{4} + b_2'\sqrt{d}n - 2b_2 n > \frac{dn}{4} + b_3\sqrt{d}n,$$

where here we used the fact that $d > d_0$ and $b_3 = \frac{b_2'}{2}$. This completes the proof. ∎

We can now prove the main result of this note.

Proof of Theorem 1.5: Fix a sufficiently large positive integer d_0, and consider an (n, d, λ) graph $G = (V, E)$ with $d > d_0$, $\lambda < c_1 d^{2/3}$, and no neighborhood with more than $c_2 d^{3/2}$ edges, where $c_1, c_2 > 0$ are small absolute constants to be chosen later, and n is even.

By Theorem 3.2 there is a dense bisection cut (V^+, V^-) of G with

$$|E(V^+, V^-)| \geq \frac{dn}{4} + b_3\sqrt{d}n.$$

Fix such a partition (V^+, V^-). We proceed to show that the condition (2) holds for all $S \subseteq V$.

Without loss of generality, we may assume that $|S| \leq n/2$. Indeed, if (2) holds for S, then it holds for S^c as well, as both sides of the inequality do not change when replacing S by S^c. Moreover, we may assume that $|S| \geq (\frac{1}{2} - \frac{\lambda}{d})n$ since otherwise it follows from (3), (4) and the facts that $\lambda < c_1 d^{2/3}$ and $d > d_0$, that

$$|E(S, S^c)| \geq \frac{(d-\lambda)|S||S^c|}{n} \geq (d-\lambda)(\frac{1}{2} + \frac{\lambda}{d})|S|$$

$$= \frac{d}{2}(1 - \frac{\lambda}{d})(1 + \frac{2\lambda}{d})|S| = \frac{d}{2}(1 + \frac{\lambda}{d} - \frac{2\lambda^2}{d^2})|S|$$

$$> \frac{d}{2}(1 + \frac{\lambda}{2d})|S| \geq \frac{d}{2}(1 + \frac{1}{2\sqrt{d}})|S| = (\frac{d}{2} + \frac{\sqrt{d}}{4})|S| \geq (\frac{d}{2} + \frac{\sqrt{d}}{4})|\ |S \cap V^+| - |S \cap V^-|\ |,$$

supplying the desired inequality. Hence, it suffices to consider sets S with $(\frac{1}{2} - \frac{\lambda}{d})n \leq |S| \leq n/2$.

Without loss of generality, we may assume that $|S \cap V^+| \geq |S \cap V^-|$. Suppose, first, that $|S \cap V^-| \geq \frac{\lambda}{d}n$. Then by (3)

$$|E(S, S^c)| \geq \frac{(d-\lambda)|S||S^c|}{n} \geq \frac{d}{2}|S| - \frac{\lambda}{2}|S| \geq (\frac{d}{2} + \frac{\lambda}{2})|S| - \lambda|S|$$

$$\geq (\frac{d}{2} + \frac{\lambda}{2})|S| - \frac{\lambda}{2}n > (\frac{d}{2} + \frac{\lambda}{2})(|S| - 2|S \cap V^-|)$$

$$= (\frac{d}{2} + \frac{\lambda}{2})(|\ |S \cap V^+| - |S \cap V^-|\ |) \geq (\frac{d}{2} + \frac{\sqrt{d}}{2})(|\ |S \cap V^+| - |S \cap V^-|\ |),$$

where the last inequality follows from (4). Thus condition (2) holds in this case.

It therefore remains to show that the condition holds for sets S with $|S \cap V^+| \geq (\frac{1}{2} - \frac{2\lambda}{d})n$, $|S \cap V^-| \leq \frac{\lambda}{d}n$. For such sets $|V^+ \setminus S| + |S \setminus V^+| \leq \frac{3\lambda}{d}n$ and hence one can apply Theorem 3.2 and Lemma 2.2 with $x = \frac{3\lambda}{d}n$ to get that

$$|E(S, S^c)| = \frac{d|S||S^c|}{n} + \delta(S, S^c) \geq \frac{d}{2}|S| + \delta(V^+, V^-) - 4\lambda\sqrt{xn} \geq \frac{d}{2}|S| + b_3 n\sqrt{d} - 4\lambda n\sqrt{\frac{3\lambda}{d}}$$

$$\geq \frac{d}{2}|S| + b_3 n\sqrt{d} - \frac{4\sqrt{3}(c_1 d^{2/3})^{3/2}}{\sqrt{d}} = \frac{d}{2}|S| + b_3 n\sqrt{d} - 4\sqrt{3}c_1^{3/2}\sqrt{d} > \frac{d}{2}|S| + \frac{b_3}{2}n\sqrt{d},$$

where the last inequality holds for an appropriate choice of $c_1 > 0$. Taking $c_3 = b_3$ we conclude that the last quantity is at least

$$(\frac{d}{2} + c_3\sqrt{d})|S| \geq (\frac{d}{2} + c_3\sqrt{d})|\ |S \cap V^+| - |S \cap V^-|\ |,$$

completing the proof. ■

4 Modular orientation of random regular graphs

The value of λ for random d-regular graphs has been studied extensively. A major result due to Friedman [9] is the following:

Lemma 4.1 ([9]) *For every fixed $\varepsilon > 0$ and for $G \in \mathcal{G}_{n,d}$, a.a.s.*

$$\lambda(G) \leq 2\sqrt{d-1} + \varepsilon\,.$$

Since it is easy and well known that for any fixed d, a.a.s., the random d-regular graph on n vertices does not contain two triangles sharing an edge (and hence certainly does not contain a neighborhood with $c_2 d^{3/2}$ edges), the assertion of Theorem 1.6 follows from Theorem 1.5 and Lemma 4.1.

5 Concluding remarks and open problems

- The assertion of Theorem 1.5 shows that there is an absolute positive constant a so that for all sufficiently large p, a d-regular graph with $d \geq (4p - a\sqrt{p})$ satisfying the conditions of the theorem has a mod $(2p+1)-$ orientation. In particular this holds, a.a.s., for a random regular graph of this degree. Note that such a graph is not $4p$-edge connected, as its minimum degree is smaller than $4p$. This is similar to the main result of Sudakov in [18] that asserts that as soon as the (non-regular) random graph $G(n,p)$ has minimum degree 2, it has, a.a.s., a nowhere-zero 3-flow (although it is obviously not 4-edge connected.)

- The proof of Theorem 1.3 here holds only for $p > p_0$ for some fixed p_0, and we have made no serious attempts to optimize its value (or optimize the constants in Theorem 1.5). This can be done but will make the computation more tedious, and will not lead to a proof that works for all values of p. It will be interesting to formulate and prove a version of the theorem for $p = 1$, which corresponds to the Conjecture of Tutte mentioned as Conjecture 1.1 here. For the special case of random 5-regular graphs this has been proved very recently by the second author and Wormald [16].

Acknowledgment: We thank an anonymous referee for suggestions that improved the presentation.

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