CLEANING WITH BROOMS

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Abstract. A model for *cleaning* a graph with brushes was recently introduced. Most of the existing papers consider the minimum number of brushes needed to clean a given graph G in this model, the so-called *brush number* $b(G)$. In this paper, we focus on the *broom number*, $B(G)$, that is, the maximum number of brushes that can be used to clean a graph G in this model.

1. Introduction

The *cleaning model* for searching, or rather decontaminating, a graph G was introduced in [9, 10]. Each vertex has a number of brushes assigned to it and at time $t = 0$, all the edges and vertices are declared *dirty*. On each tick of the clock, a dirty vertex, x, is chosen, if one exists, that has at least as many brushes as dirty incident edges, then x sends exactly one brush along each edge incident to x. The vertex x and all its incident edges are declared *clean* and those clean edges could be regarded as being deleted from the graph. The first part of the problem is to determine an initial configuration of brushes and a sequence of vertices which, when cleaned in that order, will result in every vertex and edge being cleaned. Note that a dirty vertex may have no dirty incident edges but it still needs to be cleaned. Such a sequence of vertices is called a *cleaning sequence*. The focus of [1, 4, 5, 9, 10, 11, 14] is, for a graph G, to determine the minimum number of brushes required to clean the graph, called the *brush number* and denoted $b(G)$.

In this paper, we consider the *broom number*, $B(G)$, that is, the maximum number of brushes that can be used to clean the graph in this model where every brush has to clean at least one edge. (Note that the restriction is necessary; else 'infinitely' many brushes can be used.) The broom number of a random d–regular graph was recently studied in [13] but no other properties of $B(G)$ are known. A formal definition is given in the next section for all these concepts.

To illustrate the two concepts, take a path P_n on n vertices. It is easy to see that $b(P_n) = 1$ (start with one brush on a leaf, at each step it cleans the next edge). Clearly $B(P_n)$ is at most $|E(P_n)| = n - 1$ and this occurs if each brush cleans exactly one edge in the graph. It is easy to see that P_n can actually be cleaned using $n - 1$ brushes. Consider the example with P_7 shown in Figure 1. Initially, two brushes are placed at each of vertices v_2, v_4, and v_6. In the first step, v_2 is cleaned (shown in 1); in the second step, v_4 is cleaned (shown in 2); in the third step, v_6 is cleaned (shown in 3). At this point, every edge of the graph has been cleaned; however, the cleaning algorithm, described formally in Section 2, continues to clean each of the remaining dirty vertices.

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These steps have been omitted from Figure 1 because no brushes actually move in the remaining steps of the algorithm. Consequently, $B(P_7) = |E(P_7)| = 6$.

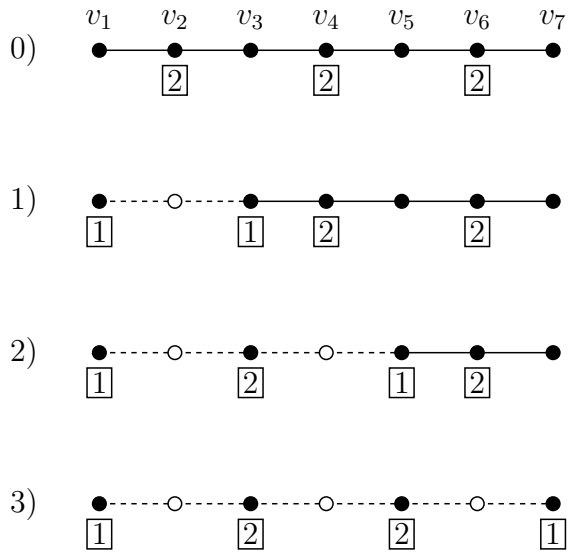

FIGURE 1. The Cleaning Process for P_7 using $B(P_7) = 6$ brushes.

Much has been discovered about the brush number. A probabilistic argument, see [1, Theorem 3.7] and Theorem 3.3, gives the average number of brushes needed to clean a graph thereby giving both an upper and a lower bound for $b(G)$ and $B(G)$, respectively. Indeed, [1, 11, 14] are all concerned with brush numbers of random graphs (both random d–regular graphs and binomial random graphs). In [9, 10] results for general and specific graphs are given. In [4], a slightly different model is used: instead of cleaning one vertex at a time (*sequential model*), all vertices that can be cleaned are (*parallel model*). For example, mechanized cleaning agents have recently been used to periodically clean a network of pipes from a regenerating contaminant, such as algae or zebra mussels (see [6, 8]) and have been suggested as a means of removing biofilm in cases where chemical or biological controls are unavailable or infeasible as the only method or decontamination. As reported in [12], "Any kind of mechanical action will not only improve cleaning results, but will also result in increased microbial kill, when disinfectants are used." The cases presented in [6, 8] involve a nuclear power plant. Shutting down the plant whilst cleaning the pipes is not a feasible option hence the need for mechanical devices that can clean during regular operating conditions. If a device needs servicing, repair or replacement, then the pipe network and the associated plant would have to be shut down with possibly huge financial implications to the company and inconvenience to customers. One way of maximizing the periods between mechanical breakdowns is to minimize the work done by each machine. In this case, maximize the number of machines in the network without introducing unnecessary redundancies, i.e. machines which sit idle during a cleaning sequence. This is one motivation for looking at the brush number.

In our models, once the graph has been cleaned, the edges are re-introduced and declared dirty. In the sequential model, for any cleaning sequence, the final configuration of brushes is a perfectly good initial configuration that results in a second cleaning sequence (see [9, 10]) and so the graph can be continually and automatically cleaned. This is not true for parallel cleaning. (For example, take a triangle with two brushes at one vertex. The triangle can be cleaned exactly once.) However, this is not to say that sequential cleaning is easy. On the contrary in general, it is difficult to determine $b(G)$.

Brush Cleaning

Instance: A graph $G = (V, E)$ and integer $k \geq 0$.
Question: Is $b(G) \leq k$?

In [4], Brush Cleaning was translated into Balanced Vertex–Ordering, which was known from [3] to be $\mathcal{NP}$-complete. More specifically, the problem remains $\mathcal{NP}$-complete for bipartite graphs of maximum degree 6 [3], planar graphs of maximum degree 4 [7], and 5-regular graphs [7].

In the next section, we formally introduce the problem and associated concepts. In Section 3, we show that $|E(G)|/2 \leq B(G) \leq |E(G)|$ and $B(G) \leq |V(G)|^2/4$. Theorem 3.7 gives an upper bound in terms of edge decompositions which is very useful in the later sections. In Section 4, we show that adding an edge can increase the broom number by 1 or leave it unchanged, which allows us to show that for $0 \leq k \leq \lfloor \frac{n^2}{4} \rfloor$ there is a graph on n vertices with $B(G) = k$ (the lower bound increases to $n-1$ if the graph is connected). We also show that $b(G) = B(G)$ if an only if G is a disjoint union of cliques (Theorem 4.6). In Section 5, we obtain very tight bounds on the broom number of: the Cartesian product of cliques; the strong product of cycles with both cycles and cliques.

2. Definitions and previous results

The following cleaning algorithm and terminology was recently introduced in [10]. An initial configuration of brushes is denoted by $\omega_0 : V(G) \to \mathbb{N} \cup \{0\}$. At each step t, $\omega_t(v)$ denotes the number of brushes at vertex v ($\omega_t : V \to \mathbb{N} \cup \{0\}$) and D_t denotes the set of dirty vertices. An edge $uv \in E$ is dirty if and only if both u and v are dirty: $\{u, v\} \subseteq D_t$. Finally, let $E_t(v)$ denote the number of dirty edges incident to v at step t:

$$E_t(v) = \begin{cases} |N(v) \cap D_t| & \text{if } \ v \in D_t \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1. *The **cleaning process** $\mathfrak{P}(G, \omega_0) = \{(\omega_t, D_t)\}_{t=0}^{T}$ of an undirected graph $G = (V, E)$ with an **initial configuration of brushes** ω_0 is as follows:*

- **(0):** *Initially, all vertices are dirty: $D_0 = V$; set $t := 0$*
- **(1):** *Let α_{t+1} be any vertex in D_t such that $\omega_t(\alpha_{t+1}) \geq E_t(\alpha_{t+1})$. If no such vertex exists, then stop the process ($T = t$), return the **cleaning sequence** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_T)$, the **final set of dirty vertices** D_T, and the **final configuration of brushes** ω_T*

(2): *Clean α_{t+1} and all dirty incident edges by moving a brush from α_{t+1} to each dirty neighbour. More precisely, $D_{t+1} = D_t \setminus \{\alpha_{t+1}\}$, $\omega_{t+1}(\alpha_{t+1}) = \omega_t(\alpha_{t+1}) - E_t(\alpha_{t+1})$, and for every $v \in N(\alpha_{t+1}) \cap D_t$, $\omega_{t+1}(v) = \omega_t(v) + 1$, the other values of ω_{t+1} remain the same as in ω_t.*

(3): *$t := t + 1$ and go back to* **(1)**

Note that for a graph G and initial configuration ω_0, the cleaning process can return different cleaning sequences and final configurations of brushes; consider, for example, a triangle with vertices u, v, w. If we set $\omega_0(u) = 2$ and $\omega_0(v) = \omega_0(w) = 0$, then the cleaning process may clean u, v, w (leaving $\omega_2(w) = 2$) or u, w, v (leaving $\omega_2(v) = 2$). It was shown (see [10, Theorem 2.1]), however, that the final set of dirty vertices is determined by G and ω_0. Thus, the following definition is natural.

Definition 2.2. *A graph $G = (V, E)$* ***can be cleaned*** *by the initial configuration of brushes ω_0 if the cleaning process $\mathfrak{P}(G, \omega_0)$ returns an empty final set of dirty vertices ($D_T = \emptyset$).*

The brush number, $b(G)$, is the minimum number of brushes needed to clean G, that is,

$$b(G) = \min_{\omega_0 : V \to \mathbb{N} \cup \{0\}} \Big\{ \sum_{v \in V} \omega_0(v) : G \text{ can be cleaned by } \omega_0 \Big\}.$$

An equivalent formulation of the problem is useful. Fix a permutation, α of the vertices and start with zero brushes. Clean the graph in the order given by α adding the necessary brushes (if any) required for the next vertex to be cleaned. That number is equal to the difference of the number of adjacent vertices not cleaned and those already cleaned. More specifically, Let $\alpha = (x_1, x_2, \ldots, x_n)$ be a permutation of the vertices of G; for each vertex x_i let $N^+(x_i) = \{x_j : x_j x_i \in E \text{ and } j > i\}$ and $N^-(x_i) = \{x_j : x_j x_i \in E \text{ and } j < i\}$; finally let

$$b_\alpha(G) = \sum_{i=1}^{n} \max\{|N^+(x_i)| - |N^-(x_i)|, 0\}. \tag{1}$$

The *brush* number is given by

$$b(G) = \min_{\alpha} b_\alpha(G).$$

It is clear that for every cleaning sequence α, $b_\alpha(G) \geq b(G)$ and that $b_\alpha(G)$ only counts brushes that actually clean at least one edge. In this paper we focus on the worst-case scenario; that is, we would like to determine the cleaning sequence which uses as many brushes as possible. This, of course, gives an upper bound for any cleaning sequence.

Definition 2.3. *The broom number, $B(G)$, of a given graph $G = (V, E)$ is*

$$B(G) = \max_{\alpha} b_\alpha(G).$$

For example, we may clean $C_8 = (v_1, v_2, \ldots, v_8)$ with only two brushes, using cleaning sequence $\gamma = (v_1, v_2, \ldots, v_8)$. That is, $b_\gamma(C_8) = 2$ (in fact, $b(C_8) = 2$). However, we could also clean C_8 with eight brushes, using cleaning sequence $\alpha = (v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_8)$.

That is, $b_\alpha(C_8) = 8$ (see Figure 2 for the initial configuration of brushes). Clearly the maximum number of brushes one could use to clean any graph G is $|E(G)|$: each brush cleans exactly one edge. Consequently, $B(C_8) = b_\alpha(C_8) = 8$.

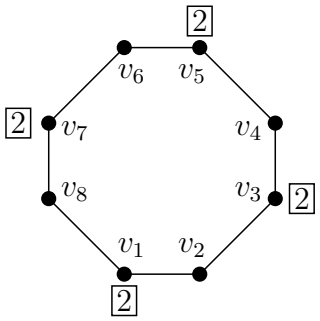

FIGURE 2. An initial configuration to clean C_8 using $B(C_8) = 8$ brushes.

The model presented in this paper is such that once every vertex and edge of a graph have been cleaned, the vertices and edges will become re-contaminated (continually), say by algae, so that cleaning is regarded as an on-going process. Ideally, the final configuration of the brushes, after all the edges have been cleaned, should be a viable starting configuration to clean the graph again. This is always possible: the following theorem has been proven in [10] (Theorem 2.3), although the statement presented here is a bit stronger focusing on the cleaning sequence that can be used. Thus, if we can clean a graph *once* using $b_\alpha(G)$ brushes and afterward the brushes remain at their respective final configurations, then if the vertices and edges become re-contaminated, we may clean the graph using the same $b_\alpha(G)$ brushes again.

Theorem 2.4 ([10])**.** ***The Reversibility Theorem***
Given the initial configuration ω_0, suppose G can be cleaned using cleaning sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and yielding final configuration ω_n, $n = |V(G)|$. Then, given initial configuration $\tau_0 = \omega_n$, G can be cleaned using cleaning sequence $\overline{\alpha} = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$ and yielding the final configuration $\tau_n = \omega_0$. Moreover, $b_\alpha(G) = b_{\overline{\alpha}}(G)$.

When a graph G is cleaned using the cleaning process described in Definition 2.1, each edge of G is traversed exactly once and by exactly one brush. Note that no brush may return to a vertex it has already visited, motivating the following definition.

Definition 2.5. *The **brush path** of a brush b is the path formed by the set of edges cleaned by b.*

3. General bounds

Let us start with the following results that provide an upper and lower bound for the broom number of a general graph.

By definition, G can be decomposed into $b_\alpha(G)$ brush paths. After a graph has been cleaned using the minimal brush configuration, note that no brush can stay at its initial vertex: consequently, these brush paths each contain at least one edge. Thus, the maximum number of paths into which a graph G can be decomposed (that is, the number of edges) yields an upper bound for $B(G)$.

Observation 3.1. *For any graph $G = (V, E)$, $B(G) \leq |E|$.*

This upper bound can be obtained if G is bipartite. We will show that the broom number is smaller than this trivial bound otherwise.

Theorem 3.2. *$B(G) = |E|$ if and only if G is bipartite.*

Proof. Let $G = (V = X \cup Y, E)$ be a bipartite graph with partite sets X and Y. First, we clean the vertices in X (in any order). Once every vertex in X has been cleaned, every edge of G has been cleaned. Note that it is not possible to reuse any brush, so the number of brush paths we start up to this point of the process is $|E|$. We then clean the vertices in Y (again, the order is not important) to clean the graph. Combining this result with Observation 3.1, we find that $B(G) = |E|$.

Suppose now that G is not bipartite. Consider any cleaning sequence α and vertices $v_1, v_2, \ldots, v_{2k+1}$ ($k \in \mathbb{N}$) that induce an odd cycle C. It is clear that at some point in the cleaning process, we clean vertex v_i ($i \in [2k + 1]$) with the property that one of its neighbours in C has already been cleaned (say, v_{i-1}) and the other has not (v_{i+1}) as shown in Figure 3. Clearly there is some brush which has traversed the edge $v_{i-1}v_i$ from v_{i-1} to v_i. When v_i is cleaned, that brush can now traverse a second edge and so that brush path is of length at least two. This implies that at least one brush path has length at least two and $B(G) \leq |E| - 1$.

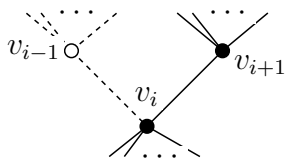

FIGURE 3. The step at which vertex v_i is to be cleaned in the proof of Theorem 3.2.

□

Now, let us move to the lower bound. An argument that provides an upper bound for the brush number of a general graph (see [1, Theorem 3.7]) can be easily modified to obtain a lower bound for the broom number.

Theorem 3.3.

$$B(G) \geq \frac{|E|}{2} + \frac{|V|}{4} - \frac{1}{4} \sum_{v \in V(G), \deg(v) \text{ is even}} \frac{1}{\deg(v) + 1}$$

for any graph $G = (V, E)$.

Proof. Let π be a random permutation of the vertices of G taken with uniform distribution. We clean G according to this permutation to get the value of $b_\pi(G)$ (note that $b_\pi(G)$ is a random variable now). For a vertex $v \in V$, it follows from (1) that we have to assign to v exactly

$$X(v) = \max\{|N^+(v)| - |N^-(v)|, 0\} = \max\{2|N^+(v)| - \deg(v), 0\}$$

brushes in the initial configuration, where $|N^+(v)|$ is the number of neighbors of v that follow it in the permutation (that is, the number of dirty neighbours of v at the time when v is cleaned). The random variable $|N^+(v)|$ attains each of the values $0, 1, \ldots, \deg(v)$ with probability $1/(\deg(v)+1)$; indeed, this follows from the fact that the random permutation π induces a uniform, random permutation on the set of $\deg(v)+1$ vertices consisting of v and its neighbors. Therefore the expected value of $X(v)$, for even $\deg(v)$, is

$$\frac{\deg(v) + (\deg(v)-2) + \cdots + 2}{\deg(v)+1} = \frac{\deg(v)+1}{4} - \frac{1}{4(\deg(v)+1)},$$

and for odd $\deg(v)$ it is

$$\frac{\deg(v) + (\deg(v)-2) + \cdots + 1}{\deg(v)+1} = \frac{\deg(v)+1}{4}.$$

Thus, by linearity of expectation,

$$\begin{aligned} \mathbb{E}b_\pi(G) &= \mathbb{E}\left(\sum_{v\in V} X(v)\right) = \sum_{v\in V} \mathbb{E}X(v) \\ &= \frac{|E|}{2} + \frac{|V|}{4} - \frac{1}{4} \sum_{v\in V(G), \deg(v) \text{ is even}} \frac{1}{\deg(v)+1}, \end{aligned}$$

which means that there is a permutation π_0 such that $B(G) \geq b_{\pi_0}(G) \geq \mathbb{E}b_\pi(G)$ and the assertion holds. □

A slightly weaker bound than the one presented in Theorem 3.3 can be obtained as follows. Denote by $maxcut(G)$ the maximum size of an edge cut in $G = (V, E)$. Let $V = A \cup B$ be a partition of V with $|E(A, B)| = maxcut(G)$. Define a permutation $\pi = (v_1, v_2, \ldots, v_n)$ of V by putting the vertices of A first in an arbitrary order, followed by the vertices of B in an arbitrary order. Now, let us clean graph G using permutation π. It is clear that after cleaning the vertices of A exactly one brush is sent through every edge between A and B. Moreover, these brushes cannot be reused (at least, not at this point). Thus we get the following.

Observation 3.4. *For any graph $G = (V, E)$, $B(G) \geq maxcut(G)$.*

Now, using the well-known fact that every graph G has a bipartite subgraph with at least half the edges of G (see, for example, the textbook [2] on the probabilistic method) we get that $B(G) \geq |E|/2$.

Let us also note that slightly improved bound can be obtained for triangle free graphs. In order to do that, one can apply the result of Shearer [15] who showed that a triangle-free graph $G = (V, E)$ with degree sequence $(d_1, d_2, \ldots, d_n)$ has a cut of size at least $|E|/2 + c\sum_{i=1}^{n} \sqrt{d_i}$ for some (explicit) constant $c > 0$. In particular, it follows from Theorem 3.3 that for a d-regular graph on n vertices $B(G) \geq \frac{n}{4}(d + \Omega(1))$, whereas for triangle free graphs we get that $B(G) \geq \frac{n}{4}\left(d + \Omega(\sqrt{d})\right)$.

From this we get immediately the following corollary.

Corollary 3.5. *For any graph $G = (V, E)$ with $E \neq \emptyset$,*

$$\frac{1}{2} \leq \frac{B(G)}{|E|} \leq 1.$$

Note also that the bound in Theorem 3.3 is tight when G is a union of cliques (since all cleaning sequences are equivalent in this case). In fact

$$B(K_n) = b(K_n) = \left\lfloor \frac{n^2}{4} \right\rfloor \tag{2}$$

for any complete graph K_n (see [10, Theorem 5.2]). On the other hand, the difference between $B(G)$ and $b(G)$ can be arbitrarily large. For example $B(P_n) = n - 1$ whereas $b(P_n) = 1$, or $B(K_{m,n}) = mn$ (see Theorem 3.2) and $b(K_{m,n}) = \lceil mn/2 \rceil$ (shown in [9]; see also [5] where the parallel version of the cleaning process was used to clean $K_{m,n}$).

Observation 3.1 provides an upper bound in terms of the number of edges. In the following theorem, the upper bound is a function of the number of vertices. Again, the result is sharp: the upper bound is obtained for K_n and $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Theorem 3.6. *For any graph G on n vertices,*

$$B(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even;} \\ \\ \frac{n^2-1}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Before we move to the proof of this theorem, let us mention that it is a simple implication of the Lemma 4.1 proved in the next section. We present an alternative proof below.

Proof of Theorem 3.6. We use induction on the number of vertices n. The base case ($n = 1$) is obvious.

First, suppose n is even. For inductive step assume that $B(G) \leq n^2/4$ for all graphs on n vertices. Let G_{n+1} be an arbitrary graph on $n+1$ vertices. We would like to show that

$$B(G_{n+1}) \leq n^2/4 + n/2.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be a cleaning sequence that uses $B(G_{n+1})$ brushes to clean G_{n+1} and let ω_0 and ω_{n+1} be the initial configuration and, respectively, the final configuration associated with α.

Suppose first that there exists vertex v in G_{n+1} with $\omega_0(v) = \omega_{n+1}(v) = 0$. Create a new graph G' from G_{n+1} by removing vertex v and creating $\deg(v)/2$ new edges from vertices in $N^+(v)$ to vertices in $N^-(v)$ such that $\deg_{G'}(u) = \deg_{G_{n+1}}(u)$ for each $u \in V(G')$. Note that G' does not have to be a simple graph; let G_n be a simple graph obtained from G' by replacing multiple edges by a single edge. As $\deg(v)/2 \leq n/2$, there were at most $n/2$ brush paths passing through vertex v in G_{n+1} and so at most $n/2$ multiple edges were created in G'. Consequently,

$$B(G_{n+1}) \leq B(G_n) + n/2 \leq n^2/4 + n/2$$

by the inductive hypothesis.

Suppose now that there is no vertex in G_{n+1} with $\omega_0(v) = \omega_{n+1}(v) = 0$, that is, each vertex starts or finishes at least one brush path if the cleaning sequence α is used. Without loss of generality, we can assume that there are at most $n/2$ vertices with $\omega_0(v) = 0$ (if this is not the case, then $\omega_{n+1}(v)$ has this property and $\omega_{n+1}(v)$ and $\overline{\alpha}$, the reverse sequence of α, can be considered instead of $\omega_0(v)$ and α). If we delete an edge from α_1 (the first vertex cleaned) to a vertex α_i with $\omega_0(\alpha_i) = 0$ (at least one brush gets stuck in α_i; otherwise $\omega_{n+1}(\alpha_i) = 0$), we can remove one brush from α_1 in the initial configuration in order to still be able to clean a graph using the cleaning sequence α. On the other hand, if an edge from α_1 to vertices with $\omega_0(\alpha_i) > 0$ is removed, then this does not affect $b_\alpha(G)$ (one brush has to be moved from α_1 to α_i). Thus, if we delete vertex α_1 together with all its incident edges, we save at most $n/2$ brushes and still be able to clean $G_n = G_{n+1} \setminus \alpha_1$ using the cleaning sequence $\widetilde{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_{n+1})$. Therefore,

$$B(G_{n+1}) \leq b_{\widetilde{\alpha}}(G_n) + n/2 \leq B(G_n) + n/2 \leq n^2/4 + n/2$$

by inductive hypothesis.

The proof for n odd is exactly the same. We assume that $B(G) \leq n^2/4 - 1/4$ for all graphs on n vertices and show that $B(G_{n+1}) \leq n^2/4 + n/2 + 1/4$. If there is a vertex with $\omega_0(v) = \omega_{n+1}(v) = 0$, we get $B(G_{n+1}) \leq B(G_n) + (n-1)/2 \leq n^2/4 + n/2 - 3/4$; otherwise, we get $B(G_{n+1}) \leq B(G_n) + (n+1)/2 \leq n^2/4 + n/2 + 1/4$ and the assertion follows. □

We finish this section with an upper bound of $B(G)$ in terms of an edge decomposition of G which will be very useful in determining the broom number of products of graphs we consider in Section 5.

Theorem 3.7. *Given a graph $G = (V, E)$, let $E_1, E_2, \ldots, E_k$ be any partition of E and let $G_i = (V, E_i)$ for $i \in \{1, 2, \ldots, k\}$. Then*

$$B(G) \leq \sum_{i=1}^{k} B(G_i).$$

Proof. Consider the cleaning sequence α that yields $B(G)$: that is, $B(G) = b_\alpha(G)$. This cleaning sequence can be used to clean each subgraph G_i ($i \in \{1, 2, \ldots, k\}$) to get a decomposition of E_i containing $b_\alpha(G_i)$ brush paths $\{b_i^1, b_i^2, \ldots, b_i^{b_\alpha(G_i)}\}$. Now, we can clean the original graph G using cleaning sequence α, but each time a brush traverses the first edge of any brush path b_i^j ($i \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, b_\alpha(G_i)\}$), we assign

this brush to b_i^j and the brush must follow the path to its last vertex; after this the brush can be reused. It is clear that this is a valid cleaning process of G (the shape of brush paths depends on b_i^j's however) that yields $b_\alpha(G)$ brush paths. Each brush path consists of one or more b_i^j's which implies that

$$B(G) = b_\alpha(G) \leq \sum_{i=1}^{k} b_\alpha(G_i) \leq \sum_{i=1}^{k} B(G_i).$$

This finishes the proof of the theorem. □

4. Other results

In this section we investigate which values for the broom number can be obtained when a graph on n vertices is cleaned. For any graph G with $e \notin E(G)$, it was shown in [9] that

$$b_\alpha(G) - 1 \leq \quad b_\alpha(G + e) \quad \leq b_\alpha(G) + 1 \tag{3}$$

for any cleaning sequence α. This implies that adding an edge can only change the brush number by 1. It is easy to construct examples to show that both inequalities are tight. In other words, there are pairs (G, e) such that the brush number decreases, stays the same, or increases when the edge e is added to the graph G. A similar, but slightly stronger, property holds for the broom number, namely, the broom number never decreases with the addition of an edge.

Lemma 4.1. *For any graph G with $e \notin E(G)$,*

$$B(G) \leq B(G + e) \leq B(G) + 1$$

From the lemma we get immediately the following corollary.

Corollary 4.2. *For any two graph G, H such that $G \subset H$ we have $B(G) \leq B(H)$.*

In order to prove Lemma 4.1 we need the following useful property of a cleaning sequence that yields $B(G)$. It is clear that in order to maximize the number of brushes used there is no point to clean a vertex 'for free' when there is another dirty vertex available for which we need to introduce brushes in the initial configuration. Therefore, one can assume that that the cleaning sequence α yielding $B(G)$ has the property that $|N^+(\alpha_i)| - |N^-(\alpha_i)|$ is positive up to some value of i; the remaining terms are non-positive. In fact, the following (a bit stronger) result was proved in [13].

Lemma 4.3. *For any graph $G = (V, E)$, there is a cleaning sequence α yielding $B(G)$ which is sorted with respect to $|N^+(\alpha_i)| - |N^-(\alpha_i)|$, that is, $|N^+(\alpha_i)| - |N^-(\alpha_i)| \geq |N^+(\alpha_{i+1})| - |N^-(\alpha_{i+1})|$ for $1 \leq i \leq |V| - 1$.*

Proof of Lemma 4.1. The upper bound follows immediately from (3): for every cleaning sequence α, $b_\alpha(G+e) \leq b_\alpha(G)+1 \leq B(G)+1$. It remains to show that $B(G+e) \geq B(G)$.

Consider the cleaning sequence α that uses $b_\alpha(G) = B(G)$ brushes. Based on Lemma 4.3, we can assume that α is sorted with respect to $|N^+(\alpha_i)| - |N^-(\alpha_i)|$. Now, let us add a new edge $e = \alpha_x\alpha_y$ with $x < y$ to the graph G. Since $B(G+e) \geq b_\alpha(G+e)$, it is enough to show that $b_\alpha(G + e) \geq B(G) = b_\alpha(G)$.

The two sums we consider (see (1)) for $b_\alpha(G+e)$ and for $b_\alpha(G)$, respectively, can differ in at most two terms: for $i=x$ and $i=y$. Since, after adding edge e, α_x has one more dirty neighbour when it is cleaned, the term for $i=x$ cannot decrease. If the term for $i=x$ after adding e increases by 1 (we need to 'pay' more to clean α_x), then the assertion follows even if the term for $i=y$ decreases by 1. If the term for $i=x$ remains unchanged (we clean α_x 'for free' before as well as after adding e), then the other term must remain unchanged, since the cleaning sequence is sorted. □

Now, we are ready to prove which values can be obtained for the broom number. Since exactly the same argument can be used for the brush number, we consider both numbers in the following theorem. Note that the proof is non-constructive; we do not know how to explicitly construct a graph on n vertices with a given brush number. This remains an open problem. Constructing a graph with a given broom number is much easier. Indeed, one can consider a bipartite graph G with partite sets containing $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices, respectively. If $|E(G)| = k$, then $B(G) = k$.

Theorem 4.4. *Fix any integer $n \geq 1$. Then for each $k = 0, 1, \ldots, \lfloor n^2/4 \rfloor$, there exist graphs G and G' on n vertices with $B(G) = b(G') = k$. No other value can be obtained.*

Proof. Consider a sequence $\{G_t = ([n], E_t)\}_{0 \leq t \leq \binom{n}{2}}$ of simple graphs on n vertices. Let G_0 be the empty graph. For $t \geq 1$ we form G_t from G_{t-1} by adding *any* edge; $G_{\binom{n}{2}}$ is a clique. Now, consider two sequences of numbers $\{B(G_t)\}_{0 \leq t \leq \binom{n}{2}}$ and $\{b(G_t)\}_{0 \leq t \leq \binom{n}{2}}$; $B(G_0) = b(G_0) = 0$ and $B(G_{\binom{n}{2}}) = b(G_{\binom{n}{2}}) = \lfloor n^2/4 \rfloor$. Moreover, it follows from (3) and Lemma 4.1 that the two consecutive numbers can differ by at most one. Therefore, there exist t and t', $0 \leq t, t' \leq \binom{n}{2}$ with $B(G_t) = b(G_{t'}) = k$ for each $k, 0 \leq k \leq \lfloor n^2/4 \rfloor$.

Observation 3.1 ensures that no other value for the broom number can be obtained. (The same conclusion can be deducted from the fact that $\{B(G_t)\}_{0 \leq t \leq \binom{n}{2}}$ is non-decreasing based on Lemma 4.1.) Since

$$b(G) \leq \frac{|E|}{2} + \frac{|V|}{4} - \frac{1}{4} \sum_{v \in V(G), \deg(v) \text{ is even}} \frac{1}{\deg(v)+1} \leq \frac{n^2}{4}$$

(see [1, Theorem 3.7]), the same property holds for the brush number. □

It is clear that not all possible values of the broom number can be obtained if an additional condition for a graph is added. For example, $B(G)$ cannot be too small if G is assumed to be connected. However, we can start with a path on n vertices for which $B(P_n) = |E(P_n)| = n-1$ (since P_n is bipartite) and repeat the argument we used before to get that all values between $n-1$ and $\lfloor n^2/4 \rfloor$ can be obtained. This covers all possible values by Lemma 4.1.

Observation 4.5. *For any connected graph G on n vertices, $B(G) \geq n-1$. Moreover, for each $k = n-1, n, \ldots, \lfloor n^2/4 \rfloor$, there exists a connected graph G on n vertices with $B(G) = k$. No other value can be obtained.*

Now, we would like to characterize those graphs that have the property that no matter which cleaning sequence is used, each one requires the same number of brushes. In other words, we would like to determine when $B(G) = b(G)$. As mentioned, cliques have this property and it is not difficult to see that a disjoint union of cliques also satisfies the required property. The following theorem shows this covers all possibilities.

Theorem 4.6. *Let G be a graph. Then $B(G) = b(G)$ if and only if G is a disjoint union of cliques.*

Proof. Note that the order in which vertices are cleaned in one connected component does not affect any other component, that is, the brush number of a graph G is a sum of the brush numbers of all connected components of G. Thus, without loss of generality, we can assume that G is connected. Since $B(K_n) = b(K_n) = \lfloor n^2/4 \rfloor$, it remains to show that $B(G) > b(G)$ if G is not a clique.

Fix a cleaning sequence α and let $f_\alpha(v) = |N^+_\alpha(v)| - |N^-_\alpha(v)|$. Then following (1),

$$b_\alpha(G) = \sum_{v \in V(G)} \max\{f_\alpha(v), 0\}.$$

Suppose there exist adjacent vertices v, u that are cleaned consecutively (v is cleaned first). If u, v are exchanged (the rest of the sequence remains unchanged), then $f_\alpha(v)$ decreases by 2 whereas $f_\alpha(u)$ increases by 2. Thus, if we can find a cleaning sequence α with two consecutive vertices v, u connected by an edge, and one of the following conditions holds, then the proof is finished (just swap those vertices and the brush number is going to change):

(A) $f_\alpha(u) \geq 0$ and $f_\alpha(v) \leq 1$,
(B) $f_\alpha(u) = -1$ and $f_\alpha(v) \neq 1$,
(C) $f_\alpha(u) \leq -2$ and $f_\alpha(v) \geq 1$.

Since the graph is connected but is not a clique, there exist three vertices u, v, z that induce a path $P = (u, v, z)$. Consider a cleaning sequence $(v, u, z, \alpha_1, \alpha_2, \ldots, \alpha_{n-3})$ where v is cleaned first, then u and z; the order of other vertices is not important. Since $\deg(u) \geq 1$, $f_\alpha(u) \geq -1$. Now, move some neighbours of u (if it is necessary) in front of the triple (v, u, z) to get $f_\alpha(u) \in \{-1, 0\}$ (note that z stays after u). We have a few cases to consider:

- if $f_\alpha(u) = 0$ and $f_\alpha(v) \leq 3$, then move z before v to get $f_\alpha(v) \leq 1$ (note that $f_\alpha(u)$ does not change); we have (A),
- if $f_\alpha(u) = 0$ and $f_\alpha(v) \geq 4$, then move any neighbour of u before v to get $f_\alpha(u) = -2$ and $f_\alpha(v) \geq 2$; we have (C),
- if $f_\alpha(u) = -1$ and $f_\alpha(v) = 1$, then move z before v to get $f_\alpha(v) = -1$; we have (B),
- if $f_\alpha(u) = -1$ and $f_\alpha(v) \neq 1$, then (B) can be immediately used.

□

5. Graph products

The **Cartesian product** of graphs G and H, written $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where $(u, v) \in V(G \square H)$ is adjacent to $(u', v') \in V(G \square H)$

when either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. The **strong product** of graphs G and H, written $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ where $(u, v) \in V(G \boxtimes H)$ is adjacent to $(u', v') \in V(G \boxtimes H)$ when either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$ or $vv' \in E(H)$ and $uu' \in E(G)$.

We start with the Cartesian product of cliques.

Theorem 5.1. *Fix any integer $n \geq 2$. Then*

$$\frac{n^3}{2} - \frac{n^2}{4} - \frac{n}{8} \leq B(K_n \square K_n) \leq \begin{cases} \frac{n^3}{2} & \text{if } n \text{ is even} \\ \\ \frac{n^3}{2} - \frac{n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

In particular, $B(K_n \square K_n) = (1 + o(1))n^3/2$.

Proof. We first note that $K_n \square K_n$ contains $2n$ edge disjoint cliques of size n. From (2), we know that $B(K_n) = \lfloor n^2/4 \rfloor$. Using Theorem 3.7,

$$\begin{aligned} B(K_n \square K_n) &\leq \sum_{i=1}^{2n} B(K_n) = 2n\lfloor n^2/4 \rfloor \\ &= \begin{cases} n^3/2 & \text{if } n \text{ is even} \\ n^3/2 - n/2 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The lower bound comes from Theorem 3.3. Since $K_n \square K_n$ is d–regular with $d = 2(n-1)$, we get that

$$\begin{aligned} B(K_n \square K_n) &\geq \frac{n^2}{4}\left(d + 1 - \frac{1}{d+1}\right) = \frac{n^2}{4}\left(2n - 1 - \frac{1}{2n-1}\right) \\ &= \frac{n^3}{2} - \frac{n^2}{4} - \frac{n}{8} - \frac{1}{16} - \frac{1}{32n - 16} \end{aligned}$$

and the assertion follows provided that $n \geq 2$. □

The proof of Theorem 5.1 can be easily generalized to an asymmetric case so it is omitted.

Theorem 5.2. $B(K_m \square K_n) = (1 + o(1))(m + n)mn/4$.

Next, we focus on the strong product, beginning with the product of two cycles.

Theorem 5.3. *Fix any integers $m, n \geq 3$. Then*

$$B(C_m \boxtimes C_n) = 3mn,$$

if at least one of m, n is even; otherwise (that is, if both m and n are odd)

$$3mn - 2\min\{m, n\} - 1 \leq B(C_m \boxtimes C_n) \leq 3mn.$$

In particular, $B(C_m \boxtimes C_n) = (1 + o(1))3mn$.

Proof. The upper bound follows from the fact that $C_m \boxtimes C_n$ consists of mn edge disjoint copies of a graph H on 4 vertices, the triangle with an isolated edge attached. It is clear that $B(H) = 3$ (clean vertex of degree 3 first). Theorem 3.7 yields

$$B(C_m \boxtimes C_n) \leq mnB(H) = 3mn.$$

While the non-constructive, probabilistic result (Theorem 3.3) gives a lower bound of $2mn$, it is not difficult to construct the cleaning sequence that requires much more brushes. Let $C_n = (v_1, v_2, \ldots, v_n)$ and $C_m = (w_1, w_2, \ldots, w_m)$.

Suppose first that both m and n are even. Consider the following 3 phases that can be used to clean the graph:

- $8\frac{mn}{4}$ brushes are required to clean vertices (v_{2i}, w_{2j}), $i \in \{1, 2, \ldots, n/2\}$, $j \in \{1, 2, \ldots, m/2\}$. (Note that those vertices form an independent set so the order of cleaning is not important and we need exactly 8 brushes per one vertex cleaned.)
- $4\frac{mn}{4}$ brushes are required to clean vertices (v_{2i-1}, w_{2j}), $i \in \{1, 2, \ldots, n/2\}$, $j \in \{1, 2, \ldots, m/2\}$ since each vertex received 2 brushes from neighbours cleaned in the previous phase. (Again, vertices form an independent set.)
- Remaining vertices are cleaned 'for free'.

The total number of brushes required is $3mn$.

Suppose now that exactly one of m, n is even. Without loss of generality, we can assume that n is odd and m is even. Consider the following 5 phases that can be used to clean the graph:

- $8\frac{(n-1)m}{4}$ brushes are required to clean vertices (v_{2i}, w_{2j}), $i \in \{1, 2, \ldots, (n-1)/2\}$, $j \in \{1, 2, \ldots, m/2\}$.
- $4\frac{(n-3)m}{4}$ brushes are required to clean vertices (v_{2i+1}, w_{2j}), $i \in \{1, 2, \ldots, (n-3)/2\}$, $j \in \{1, 2, \ldots, m/2\}$.
- $6\frac{m}{2}$ brushes are required to clean vertices (v_1, w_{2j}), $j \in \{1, 2, \ldots, m/2\}$.
- $4\frac{m}{2}$ brushes are required to clean vertices (v_n, w_{2j}), $j \in \{1, 2, \ldots, m/2\}$.
- Remaining vertices are cleaned 'for free'.

Again, the total number of brushes required is $3nm$.

Finally, suppose that both n and m are odd. Without loss of generality, we can assume that $n < m$. Consider the following 6 phases that can be used to clean the graph:

- $8\frac{(n-1)(m-1)}{4}$ brushes are required to clean vertices (v_{2i}, w_{2j}), $i \in \{1, 2, \ldots, (n-1)/2\}$, $j \in \{1, 2, \ldots, (m-1)/2\}$.
- $4\frac{(n-3)(m-1)}{4}$ brushes are required to clean vertices (v_{2i+1}, w_{2j}), $i \in \{1, 2, \ldots, (n-3)/2\}$, $j \in \{1, 2, \ldots, (m-1)/2\}$.
- $6\frac{m-1}{2}$ brushes are required to clean vertices (v_1, w_{2j}), $j \in \{1, 2, \ldots, (m-1)/2\}$.
- $4\frac{m-1}{2}$ brushes are required to clean vertices (v_n, w_{2j}), $j \in \{1, 2, \ldots, (m-1)/2\}$.
- $2\frac{n-1}{2}$ brushes are required to clean vertices (v_{2i}, w_1), $i \in \{1, 2, \ldots, (n-1)/2\}$.
- Remaining vertices are cleaned 'for free'.

This time, the total number of brushes required is $3mn - 2n - 1$. □

Now, let us consider the strong product of cycle and clique.

Theorem 5.4. *Fix any integers $n \geq 3$ and $m \geq 1$. If n is even then*

$$B(C_n \boxtimes K_m) = m^2 n$$

and if n is odd then

$$\lfloor m^2(n - 3/4) \rfloor \leq B(C_n \boxtimes K_m) \leq m^2 n.$$

In particular, $B(C_n \boxtimes K_m) = (1 + o(1))m^2 n$.

Proof. To get an upper bound, we note that $C_n \boxtimes K_m$ consists of n edge disjoint copies of a graph H on $2m$ vertices: the union of K_m and $\overline{K}_m$ together with m^2 edges between K_m and $\overline{K}_m$. We will prove that $B(H) = m^2$ which provides the upper bound of the proof by Theorem 3.7.

Note that each time a vertex in H is cleaned (it does not matter which one), each dirty vertex in K_m receives one brush. Thus, we have to introduce $b(t) = \max\{2m+1-2t, 0\}$ new brushes at time t provided that a vertex in K_m is cleaned. On the other hand, each dirty vertex in $\overline{K}_m$ receives one brush only if vertex in K_m is cleaned. Thus, we have to introduce $\overline{b}(t) = \max\{m + 2 - 2t, 0\}$ new brushes at time t when the first vertex of $\overline{K}_m$ is cleaned; the difference between $b(t)$ and $\overline{b}(t)$ is at most $m - 1$. When the second vertex of $\overline{K}_m$ is cleaned the difference is at most $m - 3$, etc. When the last vertex of $\overline{K}_m$ is cleaned the difference can be as small as $1 - m$ but this can at least balance the difference we have when the first vertex is cleaned. The conclusion is that there are essentially only two possible cleaning sequences that yield $B(H)$: clean all vertices of K_m and then those of $\overline{K}_m$ or vice-versa. This implies that $B(H) = m^2$, which implies that $B(C_n \boxtimes K_m) \leq m^2 n$.

Since $C_n \boxtimes K_m$ contains $(3m - 1)mn/2$ edges, Theorem 3.3 gives a lower bound of $3m^2 n/4$. We propose a cleaning sequence which improves this trivial bound. Let $C_n = (v_1, v_2, \ldots, v_n)$ and $V(K_m) = \{w_1, w_2, \ldots, w_m\}$. Suppose first that n is even. We start by cleaning vertices (v_{2i}, w_1), $i \in \{1, 2, \ldots, n/2\}$; this requires $(n/2)(3m - 1)$ brushes. Next we clean vertices of the form (v_{2i}, w_2) which requires an additional $(n/2)(3m-3)$ brushes. We continue this way introducing $(n/2)(m+1)$ brushes during the round when vertices of the form (v_{2i}, w_m) are cleaned. The remaining graph can be cleaned 'for free' and the total number of brushes used is

$$\frac{n}{2}\Big((3m - 1) + (3m - 3) + \cdots + (m + 1)\Big) = m^2 n.$$

If n is odd, we proceed in a similar way cleaning vertices of the form (v_{2i}, w_k) ($i \in \{1, 2, \ldots, (n-1)/2\}$) during the kth round ($k \in \{1, 2, \ldots, m\}$). This requires $m^2(n-1)$ brushes. After the last round, the remaining graph but a clique of size $2m$ can be cleaned for free. Each vertex of K_{2m} has exactly m brushes received from vertices that are already cleaned. It is straightforward to see that it does not matter in which order we clean the clique; the number of extra brushes needed is $b(K_m) = B(K_m) = \lfloor m^2/4 \rfloor$. This finishes the proof of the theorem. □

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References

[1] N. Alon, P. Prałat, and N. Wormald, Cleaning regular graphs with brushes, *SIAM Journal on Discrete Mathematics* **23** (2008), 233–250.

[2] N. Alon and J.H. Spencer, The Probabilistic Method, Wiley, 1992 (Third Edition, 2008).

[3] T. Biedl, T. Chan, Y. Ganjali, M. Hajiaghayi, and D. Wood, Balanced vertex–orderings of graphs, *Discrete Applied Mathematics* **148**(1) (2005) 27–48.

[4] S. Gaspers, M.-E. Messinger, R. Nowakowski, and P. Prałat, Clean the graph before you draw it!, *Information Processing Letters* **109** (2009), 463–467.

[5] S. Gaspers, M.-E. Messinger, R. Nowakowski, and P. Prałat, Parallel cleaning of a network with brushes, *Discrete Applied Mathematics* **158** (5) (2009) 467–478.

[6] B. Hobbs, J. Kahabka, "Underwater Cleaning Technique Used for Removal of Zebra Mussels at the Fitzpatrick Nuclear Power Plant", *Proceedings of The Fifth International Zebra Mussel and Other Aquatic Nuisance Organisms Conference, The Sea Grant Nonindigenous Species Site*. 1995. accessed: 4 June, 2008 <http://www.sgnis.org/publicat/47.htm>

[7] J. Kára, J. Kratochvíl, D. Wood, On the Complexity of the Balanced Vertex Ordering Problem, *Discrete Mathematics & Theoretical Computer Science* **9** (1) (2007) 193–202.

[8] S. R. Kotler, E.C. Mallen, K.M. Tammus, Robotic Removal of Zebra Mussel Accumulations in a Nuclear Power Plant Screenhouse, *Proceedings of The Fifth International Zebra Mussel and Other Aquatic Nuisance Organisms Conference, The Sea Grant Nonindigenous Species Site*. 1995. accessed: 4 June 2008 <http://www.sgnis.org/publicat/113.htm>

[9] M.-E. Messinger, Methods of Decontaminating Networks, PhD Thesis, Dalhousie University, 2008.

[10] M.-E. Messinger, R. J. Nowakowski, and P. Prałat, Cleaning a network with brushes, *Theoretical Computer Science* **399** (2008), 191–205.

[11] M.-E. Messinger, R. J. Nowakowski, P. Prałat, and N. Wormald, Cleaning random d–regular graphs with brushes using a degree-greedy algorithm, *Proceedings of the 4th Workshop on Combinatorial and Algorithmic Aspects of Networking (CAAN 2007)*, Lecture Notes in Computer Science **4852**, Springer, 2007, 13–26.

[12] Bernhard Meyer, Approaches to prevention, removal and killing of biofilms, *International Biodeterioration & Biodegradation*, **51**, Issue 4, Hygiene and Disinfection, June 2003, 249–253.

[13] P. Prałat, Cleaning random d–regular graphs with Brooms, *Graphs and Combinatorics*, submitted, 17pp.

[14] P. Prałat, Cleaning random graphs with brushes, *Australasian Journal of Combinatorics* **43** (2009), 237–251.

[15] J. Shearer, A note on bipartite subgraphs of triangle-free graphs, *Random Structures Algorithms* **3** (1992), 223–226.

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