GRAPHS WITH THE N-E.C. ADJACENCY PROPERTY CONSTRUCTED FROM RESOLVABLE DESIGNS

CATHARINE A. BAKER, ANTHONY BONATO, NEIL A. MCKAY, AND PAWEŁ PRAŁAT

Abstract. Only recently have techniques been introduced that apply design theory to construct graphs with the n-e.c. adjacency property. We supply a new random construction for generating infinite families of finite regular n-e.c. graphs derived from certain resolvable Steiner 2-designs. We supply an extension of our construction to the infinite case, and thereby give a new representation of the infinite random graph. We describe a family of deterministic graphs in infinite affine planes which satisfy the 3-e.c. property.

1. Introduction

Adjacency properties of graphs have received much attention since Erdős and Rényi [8] first studied them in their pioneering work on random graphs. One such adjacency property is the n-e.c. property. For a positive integer n, a graph is *n-existentially closed* or *n-e.c.*, if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B can be empty), there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. We say that z is *correctly joined* to A and B. Hence, for all n-subsets S of vertices, there exist 2^n vertices joined to S in all possible ways. For example, a graph is 2-e.c. if for each pair of distinct vertices u and v, there are four vertices not equalling u and v joined to them in all possible ways. See Figure 1 for the unique isomorphism type of 2-e.c. graph with least possible number of vertices. For completeness, every graph is 0-e.c.

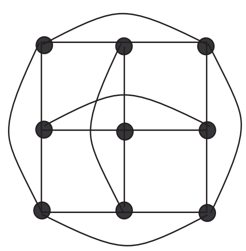

Figure 1. The smallest order 2-e.c. graph.

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Despite the fact that the n-e.c. property is straightforward to define, it is not obvious from the definition that graphs with the property exist. We say that an event holds *asymptotically almost surely* (*a.a.s.*) if the probability that it holds tends to 1 as v tends to infinity. As first proved in [8] that a.a.s. $G(v, p)$ (that is, the random graph with v vertices and fixed edge probability $p \in (0, 1)$) satisfies the n-e.c. property. Similar adjacency properties were studied by Blass and Harary [2], and in finite model theory [9, 11] to prove the zero-one law in the first-order theory of graphs.

Explicit constructions of n-e.c. graphs may be found in [3, 4], where it is proved that sufficiently large Paley graphs are n-e.c. Only recently have techniques been introduced that apply design theory to construct n-e.c. graphs. Forbes, Grannell, and Griggs [10] discovered 2- and 3-e.c. graphs arising as block intersection graphs of Steiner triple systems, while McKay and Pike [15] found 2-e.c. graphs arising from balanced incomplete block designs. Cameron and Stark [7] and independently Baker et al. [1] used randomized constructions to generate strongly regular n-e.c. graphs for all positive integers n derived from affine designs and planes, respectively. For other applications of the probabilistic method in design theory, see the survey [12].

The goal of the present paper is to give a new general construction of regular n-e.c. graphs arising from Steiner 2-designs. The methods are randomized and employ regular graphs arising from resolvable Steiner 2-designs, such as affine planes and Kirkman triple systems. Our main result is Theorem 1, which shows that a.a.s. graphs generated from certain resolvable Steiner 2-designs are n-e.c., where n tends to infinity as a logarithmic function of the number of points of the design.

If a graph is n-e.c. for all positive integers n, then the graph is called *e.c.* Any two countable e.c. graphs are isomorphic by a back-and-forth argument; the isomorphism type is named the *infinite random* or *Rado* graph, and is written R. From [8], with probability 1 a countably infinite random graph is isomorphic to R. The (deterministic) graph R has been actively studied; see [6] for a survey on R. The methods we present extend to the infinite case, giving a new representation of the infinite random graph; see Theorem 3. We finish by investigating certain deterministic graphs in infinite affine planes which satisfy the 3-e.c. property.

All graphs considered are simple, undirected, and finite unless otherwise stated. For a reference on combinatorial designs, the reader is directed to [18], while [19] is a graph theory reference. We denote the complement of the graph G by $\overline{G}$. All logarithms are in base e unless otherwise stated.

2. Random graphs from designs

A *Steiner* 2*-design* $S(2, k, v)$ is a 2-$(v, k, 1)$ design; that is, a collection of k-subsets called *blocks,* of a v-set whose elements are called *points*, such that each distinct pair of elements of this v-set is contained in a unique block. A Steiner 2-design $S(2, k, v)$ is *resolvable* if the blocks of $S(2, k, v)$ can be partitioned into sets, called *parallel classes*, so that each point of the design is contained in a unique block of each parallel class. Such a partition is called a *resolution.* A resolvable design may have more than one resolution and so we refer to a design together with a resolution as a *resolved design.* By elementary counting, there are $\frac{v-1}{k-1}$ parallel classes, so $k-1$ divides $v-1$; in addition, k divides v. For sufficiently large suitable v and choice of k, resolvable Steiner 2-designs $S(2,k,v)$ always exist, and there are infinitely many of them; see [16].

For each resolved Steiner 2-design $S = S(2,k,v)$ we may assign each parallel class a label, or *slope.* The set of slopes is denoted by L_S. Note that slopes are not points in the design, but rather identify the set of blocks from one parallel class. Given a point y of S, we define $\pi_y : S \setminus \{y\} \to L_S$ by letting $\pi_y(x)$ be the slope of the block containing x and y. By the definition of a resolved Steiner 2-design, the function π_y is well-defined. If X is a set of points in S, then let

$$\pi_y(X) = \bigcup_{x \in X} \{\pi_y(x)\}.$$

An important example of a resolvable Steiner 2-design is an *affine plane*, which is a 2-$(q^2, q, 1)$ design with q a positive integer. Affine planes are known to exist if q is a prime power, and are all resolvable. The set of slopes L_S corresponds to the points of the $(q+1)$-element line at infinity, usually written as ℓ_∞, and the function π_y is the projection function to the line at infinity. More generally, an *affine space of dimension* m is a 2-$(q^m, q, 1)$ design. We can identify the set of all the slope vectors with the points of a fixed hyperplane of codimension 1. Another example of a resolvable Steiner 2-design is a *Kirkman triple system*, which is a resolvable *Steiner triple system* (that is, a 2-$(v, 3, 1)$ design). Kirkman triple systems exist for orders v if and only if $v \equiv 3 \pmod 6$.

A design is *acceptable* if it is a resolvable Steiner 2-designs $S(2,k,v)$ with $k \leq \sqrt{v}$. It is straightforward to see that affine spaces and Kirkman triple systems are acceptable designs.

Consider an infinite set of acceptable designs, and let S be an acceptable design with v points. Fix $U \subseteq L_S$. Define $G_S(U)$ to have vertices the points of S, and two vertices p and q are joined if and only if the block containing p and q has slope in U. We will drop the subscript S if it is clear from context. Observe that the graph $G(U)$ is regular with degree $|U|(k-1)$. We note that in the case when S is an affine plane, we obtain strongly regular graphs (although this does not necessarily hold for other designs). The construction in this case was first introduced by Delsarte and Goethals, and independently by Turyn; see [17].

For example, consider the unique (up to isomorphism) Kirkman triple system of order 9:

a	b	c	d
$\{1,2,3\}$	$\{1,4,7\}$	$\{1,5,9\}$	$\{1,6,8\}$
$\{4,5,6\}$	$\{2,5,8\}$	$\{2,6,7\}$	$\{2,4,9\}$
$\{7,8,9\}$	$\{3,6,9\}$	$\{3,4,8\}$	$\{3,5,7\}$.

Here the columns consist of parallel classes with labels from $L_S = \{a, b, c, d\}$ as indicated. It is not hard to see that if we choose $U = \{a, b\}$, then $G(U)$ is isomorphic to the minimum order 2-e.c. graph in Figure 1.

We may choose U at random as follows: for a given $p \in [0, 1]$ (we allow $p = p(v)$ to be a function of the number of points), choose $m \in L_S$ to be in U independently with probability p. Hence, from an acceptable design S, we may define a probability space $\mathcal{G}(v, S, p)$ which consists of regular graphs with v vertices. Observe that G is n-e.c. if and only if $\overline{G}$ is n-e.c., and $\overline{G(U)} = G(L_S \setminus U)$. Thus, without loss of generality, we can assume that $p \leq 1/2$ since proving that a.a.s. $\mathcal{G}(v, S, p)$ is n-e.c. is equivalent to proving that a.a.s. $\mathcal{G}(v, S, 1-p)$ is n-e.c. Our main result is the following.

Theorem 1. *Let S be an acceptable design with v points, and suppose that $0 < p = p(v) \leq 1/2$. Then a.a.s. $\mathcal{G}(v, S, p)$ is n-e.c., for all $n = n(v) \leq \frac{1}{2}\log_{1/p} v - 5\log_{1/p}\log v$.*

Theorem 1 supplies a new construction of regular n-e.c. graphs. Observe that $|U|$ is a random variable with expected value $p\frac{v-1}{k-1}$. By the Chernoff bounds (see, for example, Section 2.1 of [14]), a.a.s. a graph $G \in \mathcal{G}(v, S, p)$ is regular with degree concentrated around pv, provided pv tends to infinity with v. Before we prove the theorem, we consider the following lemma.

Lemma 2. *Let S be an acceptable design with v points and X a given set of $n = n(v) \leq \log_2 v$ points from S. If v is sufficiently large, then there exists a set of points from S, written P_X, disjoint from X with the following properties.*

1. *If $q \in P_X$, then $|\pi_q(X)| = n$.*
2. *For all distinct q_1 and q_2 in P_X, $\pi_{q_1}(X) \cap \pi_{q_2}(X) = \emptyset$.*
3. *$|P_X| \geq \frac{1}{2}\sqrt{v}/\log_2^2 v$.*

Lemma 2 supplies a pool of points P_X which satisfy certain desirable independence properties. Our approach to the proof of Theorem 1 will be to prove that the probability that no vertex in P_X is correctly joined to X is sufficiently small as v becomes large.

Proof of Lemma 2. We inductively construct P_X satisfying items (1) and (2). Define $P_{X,1}$ by choosing any point $q_1 \notin X$ that is not in a block containing two points of X. For large v this eliminates at most

$$n + \binom{n}{2}k \leq \frac{kn^2}{2} \leq \frac{\sqrt{v}\log_2^2 v}{2} < v$$

points, and so we may find such a q_1.

Let $s = \lceil \frac{1}{2}\sqrt{v}/\log_2^2 v \rceil$. For a fixed $1 \leq i < s$ suppose that $P_{X,i}$ has been constructed so that $P_{X,1} \subset P_{X,2} \subset \cdots \subset P_{X,i-1} \subset P_{X,i}$, and $P_{X,i}$ satisfies (1) and (2) with $|P_{X,i}| = i$. We would like to choose $q_{i+1} \notin (X \cup P_{X,i})$ to be a point that is

i) not in a block containing two points of X, and

ii) not in a block containing a point a of X which is in a parallel class with slope in $\pi_b(P_{X,i})$ for any $b \in X$.

For $i < s$ and v large enough, conditions i) and ii) eliminate at most

$$n + \binom{n}{2}k + nik + n(n-1)ik \ \leq \ n^2 sk \ < \ v$$

points. Hence, we may find a suitable q_{i+1} satisfying items (1) and (2). Add q_{i+1} to $P_{X,i}$ to form $P_{X,i+1}$. Define $P_X = P_{X,s}$, so $|P_X| = s$ and (3) follows. □

Proof of Theorem 1. Fix finite, disjoint sets of vertices A and B in S, $|A| = a$, $|B| = b$, such that $a + b = n$, and let $X = A \cup B$. If v is sufficiently large, then there is a set P_X of vertices, disjoint from X, with cardinality $s \geq \frac{1}{2}\sqrt{v}/\log_2^2 v$ and satisfying the properties described in Lemma 2. We estimate the probability that none of the vertices of P_X is correctly joined to A and B.

By item (1) of the lemma, for x and y distinct points of X, any z in P_X has the property that the blocks containing $\{x, z\}$ and $\{y, z\}$ have distinct slopes. Note also that z and x are joined if and only if $\pi_z(x) \in U$, where U was randomly sampled with probability p from L_S. The probability that a given z in P_X is not joined correctly to A and B is therefore

$$1 - p^a(1-p)^b.$$

By item (2) of the lemma, two points of P_X induce disjoint sets $\pi_q(X)$. In particular, the events under consideration are independent. Hence, the probability that no z in P_X is correctly joined to A and B is at most

$$\left(1 - p^a(1-p)^b\right)^s. \tag{2.1}$$

As there are $\binom{v}{b}\binom{v-b}{n-b}$ many choices for A and B with $|B| = b$, by (2.1) the probability that G is not n-e.c. is therefore at most

$$\begin{aligned}
p_n &= \sum_{b=0}^{n} \binom{v}{b}\binom{v-b}{n-b}\left(1 - p^{n-b}(1-p)^b\right)^s \\
&\leq (1+o(1))v^n \sum_{b=0}^{n} \frac{1}{b!(n-b)!} \exp\left(-p^n\left(\frac{1-p}{p}\right)^b s\right) \\
&\leq (1+o(1))\frac{v^n}{n!}\sum_{b=0}^{n}\binom{n}{b}\exp\left(-p^n s\right) \\
&= (1+o(1))\frac{(2v)^n}{n!}\exp\left(-p^n s\right).
\end{aligned}$$

Now, using the Stirling's formula $n! = (1+o(1))\sqrt{2\pi n}(n/e)^n$ we obtain that

$$\begin{aligned} p_n &\leq (1+o(1))\left(\frac{2ev}{n}\right)^n \exp\left(-p^n s\right) \\ &= (1+o(1))\exp\left(n(\log v + \log(2e) - \log n) - p^n s\right) \\ &= \exp\left(O(\log^2 v) - p^{-\frac{1}{2}\log_p v + 5\log_p \log v}\Omega(\sqrt{v}/\log^2 v)\right) \\ &= \exp\left(O(\log^2 v) - \Omega(\log^3 v)\right) \\ &= o(1), \end{aligned}$$

and the assertion follows. □

Note that Theorem 1 and Lemma 2 hold for any acceptable design S. However, if S contains blocks of constant order (as in the case for a Kirkman triple system), then the cardinality of P_X in Lemma 2 is much larger; namely, $\frac{1}{2k}v/\log_2^2 v$. This also implies that for such designs, the value of n in Theorem 1 can be increased to $\log_{1/p} v - 5\log_{1/p}\log v$. Therefore, for $p = 1/2$ the result is, in some sense, tight, since no graph with v vertices can be $(\log_2 v)$-e.c. (see below for bounds on the minimum order of an n-e.c. graph).

We note that Theorem 1 gives examples of sparse regular graphs of order v which are n-e.c. with n tending to infinity in v. For example, if

$$p = v^{-\frac{1}{\log\log v}} = \exp\left(-\frac{\log v}{\log\log v}\right) = o(1),$$

then the degree of $\mathcal{G}(v,S,p)$ concentrates on $v^{1-\frac{1}{\log\log v}} = o(v)$ and

$$n = (1+o(1))\frac{1}{2}\log\log v.$$

For a positive integer n, define $m_{ec}(n)$ to be the minimum order of an n-e.c. graph. Then $m_{ec}(1) = 4$ and $m_{ec}(2) = 9$, but no other values of this function are known. For example, $20 \leq m_{ec}(3) \leq 28$; see [5]. Results in [13] obtained using a computer search demonstrate that $m_{ec}(3) \geq 24$. Using the fact that in an n-e.c. graph with $n > 1$, the neighbour and non-neighbour sets of each vertex are $(n-1)$-e.c., we have that $m_{ec}(n) \geq 2m_{ec}(n-1)+1$. As $m_{ec}(3) \geq 24$, the assertion holds and by a simple (and so omitted recursion), we derive for $n \geq 3$ that

$$m_{ec}(n) \geq (25/8)2^n - 1.$$

On the other hand, from our construction (with $p = 1/2$) it follows that $m_{ec}(n) \leq O(n^5 2^n)$ (in fact, using the random graph $G(v,1/2)$ one has that $m_{ec}(n) = O(n^2 2^n)$). It follows that

$$\lim_{n\to\infty} m_{ec}(n)^{1/n} = 2.$$

An open problem is to determine whether

$$\lim_{n\to\infty} \frac{m_{ec}(n)}{2^n}$$

exists and, if so to find its value.

3. Representations of the infinite random graph

The probability space $G(\mathbb{N}, p)$ is defined analogously as $G(v, p)$, but with vertices now the nonnegative integers. Erdős and Rényi [8] proved that with probability 1 a graph in $G(\mathbb{N}, p)$, where $p \in (0, 1)$, is isomorphic to a unique graph, called the *infinite random graph* or *Rado graph*. The graph R is deterministic and has a rich structure; it is the unique isomorphism type of countable graphs that is n-e.c. for all $n > 0$. The graph R has many well-known *representations*; that is, constructions resulting in the isomorphism type of R. These representations are diverse, using techniques from number theory, set theory, and probability; see [6] for a survey of results on R. For completeness, we describe one such representation that uses a few tools from number theory. Let the vertices of G be the set of primes $\mathbb{P}_1$ congruent to $1 \pmod 4$. The set $\mathbb{P}_1$ is infinite by Dirichlet's theorem on primes in arithmetic progressions. Two distinct primes p and q in $\mathbb{P}_1$ are joined if p is a square $\pmod q$ or q is a square $\pmod p$. The graph G is undirected by the law of quadratic reciprocity, and the n-e.c. properties follow by the Chinese Remainder theorem and Dirichlet's theorem.

In this section, we supply a new randomized representation of R using plane affine geometry. The ideas are based on those of the previous section. We work in any fixed countable affine plane $\mathcal{A}$. In this case, the set of slopes is the point set of the line at infinity, for which we will adopt the standard notation ℓ_∞. Hence, $\pi_p : \mathcal{A} \rightarrow \ell_\infty$ is the projection from p to ℓ_∞ defined in the usual way. For $U \subseteq \ell_\infty$, define $G(U)$ to have vertices the points of $\mathcal{A}$, and two vertices p and q are joined if and only if the line pq has slope in U. As in the finite case, we may choose U at random: for fixed $p \in (0, 1)$, choose $m \in \ell_\infty$ to be in U independently with probability p; with the remaining probability, m is in the complement of U. We therefore have defined a probability space which we denote $\mathcal{G}(\mathcal{A}, p)$ consisting of graphs $G(U)$ with vertices the points of $\mathcal{A}$.

Theorem 3. *Fix $p \in (0, 1)$. With probability 1, $\mathcal{G}(\mathcal{A}, p)$ is n-e.c. for all $n > 0$, and so $G(U) \cong R$.*

The following lemma has a proof similar to the proof of Lemma 2, and so we omit the proof.

Lemma 4. *For n a positive integer and X a set of n points in $\mathcal{A}$, there exists an infinite set of points P_X in $\mathcal{A}$ distinct from X with the following properties.*

(1) *If $q \in P_X$, then $|\pi_q(X)| = n$.*
(2) *For all distinct q_1 and q_2 in P_X, $\pi_{q_1}(X) \cap \pi_{q_2}(X) = \emptyset$.*

Proof of Theorem 3. Fix finite, disjoint sets of vertices A and B in $\mathcal{A}$. Let $X = A \cup B$, and say $|X| = n$. We prove that with probability 1, there is a vertex z joined to all of A and none of B.

Let P_X be a set of points distinct from X with the properties of Lemma 4. Suppose that $|A| = a$ and $|B| = b$. The probability that a given z in P_X is not joined correctly to A and B is

$$1 - p^a(1 - p)^b.$$

The probability that no z in $\mathcal{A}$ is correctly joined to A and B is zero since

$$\lim_{m\to\infty}(1-p^a(1-p)^b)^m = 0.$$

The assertion therefore holds for fixed A and B. As a countable union of measure 0 sets is measure 0, the result follows. □

We emphasize that the above proof also works with minor changes for any countably infinite affine space (or more generally for any countably infinite resolved Steiner 2-design). An interpretation of Theorem 3 is that a randomly chosen set of slopes U gives rise to R with high probability. However, the problem of finding explicit slope sets U that give rise to n-e.c. graphs is open for $n > 3$.

We now give an explicit example of an infinite 3-e.c. graph with vertices in $\mathbb{Q}^2$ and whose slope set is the union of two intervals in the set of extended rational numbers $\mathbb{Q}\cup\{\infty\}$. We will abuse notation and refer to an interval in the extended rationals as (a, b). It is easy to check that a choice of U as a single interval gives rise to 2-e.c. but not to 3-e.c. graphs. (For example, if $U = (0, 1)$, then there is no vertex z in $G(U)$ so that z is joined to $(1, 0)$ but z is not joined nor equal to either $(0, 0)$ or $(2, 0)$.) Suppose that $\mathcal{U} = (m, n) \cup (r, s)$, where $m < n < r < s$ are rational numbers (note that we use the notation $\mathcal{U}$ to denote a slope set U which is the union of finite open intervals of rationals). This choice of slopes gives rise to an infinite graph in the plane where adjacency is determined by what may be visualized as four separate sectors emanating from each point. See Figure 2.

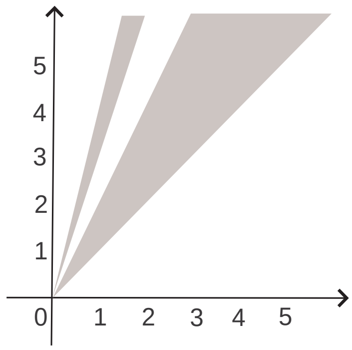

FIGURE 2. Neighbours in $G((1, 2) \cup (3, 4))$ of $(0, 0)$ in the first quadrant.

Theorem 5. *The graphs $G(\mathcal{U})$ are 3-e.c.*

Proof. Fix $\mathcal{U} = (m, n) \cup (r, s)$, where $m < n < r < s$ are rational numbers. To show that $G(\mathcal{U})$ is 3-e.c., let u, v, and w be fixed distinct points in the plane. We look for a vertex z correctly joined to u, v, and w in all of the eight cases. To simplify our discussion, we use the notation (ijk) to refer to the eight cases, where i, j, k are 0 or 1. For example, $z \notin \{u, v, w\}$ satisfies (101) if and only if it is joined to u and w, but not joined to v. We refer to (ijk) as an *adjacency problem* (or simply a *problem*) and a vertex z satisfying it as a *solution* to the problem.

There are two cases depending on whether the three points u, v, and w are collinear.

Case 1: The points u, v, and w are collinear.

Since translations in the plane preserve slopes, and rotations preserve parallelism and incidence, without loss of generality, suppose that $u = (0, 0)$, $v = (0, b)$ and $w = (0, c)$, where $0 < b < c$.

We give solutions to the eight problems (ijk) in the table below. After rotation, the set of slopes determining adjacency may include ∞; however, since none of the solutions given lies on a line with slope m, n, r or s (that is, a boundary line of a sector), we may just consider the complementary problem to find a solution.

Problem	Solution
(000)	$(0, y)$, where $y \neq 0, b, c$
(001)	$(1, y)$, where $\max(r + c, s + b) < y < s + c$
(010)	$(\frac{d}{s-m}, \frac{d(r+s)}{2(s-m)} + b)$, where $0 < d < \min(b, c - b)$
(100)	$(\frac{b}{2(n-m)}, \frac{b(n+m)}{4(n-m)})$
(011)	$(\frac{2c-b}{s-r}, \frac{(2c-b)s}{s-r} + \frac{b}{2})$
(101)	$(d, dn + \frac{(c-b)(r-n)}{r-m} + b)$, where $\max(\frac{c-b}{r-m}, \frac{(c-b)(r-n)}{(r-m)(s-n)} + \frac{b}{s-n}) < d < \frac{c-b}{r-m} + \frac{b}{r-n}$
(110)	$(\frac{b+c}{n-m}, \frac{(b+c)(m+n)}{2(n-m)})$
(111)	$(\frac{3c}{s-r}, \frac{3c(r+s)}{2(s-r)} + c)$

Case 2: The points u, v, and w are non-collinear.

As in Case 1, we may use translations and rotations so that, without loss of generality, $u = (0, 0)$, $v = (a, b)$ and $w = (0, c)$, where $0 < a$ and $0 < b < c$. In addition, since the solutions we provide do not lie on the boundaries of sectors, we need only consider the case where $\mathcal{U} = (m, n) \cup (r, s)$, which means that the vertical lines are not in $\mathcal{U}$.

Problem	Solution
(000)	at least one of $(0, b - \frac{a(n+r)}{2})$, $(0, b - \frac{a(3n+r)}{4})$, $(0, b - \frac{a(n+3r)}{4})$
(100)	(a, d), where $am < d < \min(an, am + c)$, $d \neq b$
(010)	at least one of $(0, b - \frac{a(m+n)}{2})$, $(0, b - \frac{a(3m+n)}{4})$, $(0, b - \frac{a(n+3n)}{4})$
(001)	(a, d), where $\max(ar + c, as) < d < as + c$, $d \neq b$

For the remaining problems, the relative sizes of the parameters n, b/a and $\frac{b-c}{a}$ are important. Hence, for each problem we consider various conditions on these three parameters.

Problem	Solution	Conditions
(111)	$(\frac{c+an}{n-m}, \frac{(c+an)(m+n)}{2(n-m)} + \frac{c+b-an}{2})$	$m \geq b/a$
	$(\frac{2c}{n-m}, \frac{c(3n+m)}{2(n-m)})$	$\frac{b-c}{a} < m < n \leq b/a$
	$(\frac{c+an}{n-m}, \frac{2n(c+am)}{2(n-m)} + \frac{b}{2})$	$\frac{b-c}{a} < m < b/a < n$
	$(2a + \frac{b-am}{n-m}, (2a + \frac{b-am}{n-m})(\frac{m+n}{2}) + \frac{b-am}{2})$	$\frac{b-c}{a} \geq m$
(101)	$(a, \frac{am+an+c}{2})$	$n - m > c/a$
	$(a, \frac{ar+as+c}{2})$	$s - r > c/a$
	(a, d), where d/a is any point with $r < \frac{d}{a} < s$ and $m < \frac{d-c}{a} < n$	$r - n < c/a$ and $s - m > c/a$
	$(\frac{c-b+sa}{s-r}, \frac{(c-b+sa)s}{s-r} - \frac{sa-b}{2})$	$s - r \leq c/a$ and $b/a < s$
	$(\frac{b-am}{n-m}, \frac{(b-am)m}{n-m} + \frac{c+b-am}{2})$	$s \leq b/a$ and $r - n \geq c/a$
	$(\frac{c}{r-m}, y)$, where $\max(\frac{cr}{r-m}, \frac{cn}{r-m} + b - an) < y < \min(\frac{cn}{r-m} + c, \frac{cr}{r-m} + b - ar)$, $(\frac{c}{s-n}, y)$, where $\max(\frac{cr}{s-n}, \frac{cn}{s-n} + b - an) < y < \min(\frac{cs}{s-n}, \frac{cr}{s-n} + b - ar)$, $(\frac{c}{r-m}, y)$, where $\frac{cr}{r-m} < y < \min(\frac{cs}{r-m}, \frac{cn}{r-m} + c, \frac{cm}{r-m} + b - am)$	$s \leq b/a$ and $s - m \leq c/a$ plus: $b - an < c$ and $s - r \geq n - m$, $b - an < c$ and $s - r < n - m$, $c \leq b - an$
(011)	$(\frac{ac}{2ma+c-b}, \frac{c(b-c)}{2ma+c-b} + c)$	$m < \frac{b-c}{a} < n$ or $r < \frac{b-c}{a} < s$
	$(\frac{b-ar}{s-r}, (\frac{b-ar}{s-r})s + b)$	$s \leq \frac{b-c}{a}$
	$(\frac{c+b-am}{t}, \frac{n(c+b-am)}{t} + \frac{c}{2})$, where $t = \min(r - n, n - m)$	$n < \frac{b-c}{a} \leq r$
	$(\frac{2c}{t}, \frac{2cn}{t} + \frac{b-an}{2})$, where $t = \min(r - n, n - m)$	$\frac{b-c}{a} \leq m < n < \frac{b}{a}$
	$(\frac{-c}{n-m}, \frac{-cm}{n-m} + \frac{b-am}{2})$	$\frac{b-c}{a} \leq m < \frac{b}{a} \leq n$
	$(\frac{2c}{3m-n}, \frac{4cm}{3m-n})$	$n = \frac{b-c}{a}$
(110)	$(\frac{-c}{s}, \frac{-cb}{as})$	$m < b/a < n$ or $r < b/a < s$
	$(\frac{-c}{s-r}, \frac{-cs}{s-r} + \frac{b-as+c}{2})$	$\frac{b-c}{a} < s \leq b/a$
	$(\frac{a(2c-b+as)}{ar+as-2b}, \frac{a(2c-b+as)(r+s)}{2(ar+as-2b)} + \frac{b-as}{2})$	$n < b/a < r$ and $ca(s - r) > (ar - b)(as - b)$
	$(\frac{c-b+as}{s-r}, \frac{(c-b+as)(s+r)}{2(s-r)} + \frac{b-as}{2})$	$n \leq b/a < r$ and $ca(s - r) \leq (ar - b)(as - b)$
	$(\frac{-2c}{r+s}, -c)$	$r = b/a$
	$(\frac{-2c}{4s-m-n}, \frac{-c(m+n)}{4s-m-n})$	$n = b/a$
	$(\frac{c+an-b}{n-m}, \frac{(c+an-b)(n+m)}{2(n-m)} - \frac{an-b}{2})$	$b/a \leq m$
	$(\frac{b-an}{s-n}, \frac{r(b-an)}{s-n} - \frac{c(r-n)}{s-n} + c)$	$s \leq \frac{b-c}{a}$ and $\frac{b-an}{s-n} \geq \frac{b-am}{r-m}$
	$(\frac{b-am}{r-m}, \frac{r(b-am)}{r-m} - \frac{c(r-n)(b-am)}{(b-an)(r-m)} + c)$	$s \leq \frac{b-c}{a}$ and $\frac{b-an}{s-n} < \frac{b-am}{r-m}$

□

Our final result shows that by using unions of intervals as the slope set we never obtain 4-e.c. graphs.

Theorem 6. *For all $n \geq 2$, there is no slope set $\mathcal{U}$ consisting of n disjoint open intervals (r, s), with r and s rational numbers, so that the graph $G(\mathcal{U})$ is 4-e.c.*

Proof. Consider the rationals

$$r_1 < s_1 < r_2 < s_2 < \cdots < r_n < s_n.$$

For each $1 \leq i \leq n$, let I_i denote the interval (r_i, s_i), and set

$$\mathcal{U} = \bigcup_{i=1}^{n} I_i.$$

We will show that there is a non-zero rational number t (to be chosen later in the proof) so that there is no solution to the 4-e.c. problem (1011) for the vertices $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(0, t)$.

Suppose that (a, b) is a solution to (1011) for these vertices. No point of the y-axis is joined to $(0, t)$, so $a \neq 0$. Any point of the x-axis is either joined to all three of $(-1, 0)$, $(0, 0)$, $(1, 0)$ or joined to none of them, so $b \neq 0$ also. No point $(-1, b)$ is joined to $(-1, 0)$ and no point $(1, b)$ is joined to $(1, 0)$, so $a \neq -1, 1$.

The slopes of the lines joining (a, b) to $(-1, 0)$, $(0, 0)$, $(1, 0)$ are $\frac{b}{a+1}$, $\frac{b}{a}$, $\frac{b}{a-1}$, respectively. Since $\frac{b}{a} \notin \mathcal{U}$ is always between $\frac{b}{a+1} \in \mathcal{U}$ and $\frac{b}{a-1} \in \mathcal{U}$, the slopes $\frac{b}{a+1}$ and $\frac{b}{a-1}$ must belong to distinct intervals, say $I_k = (r_k, s_k)$ and $I_m = (r_m, s_m)$, respectively.

For $1 \leq i \leq n$ let $\mathcal{L}(p, I_i)$ denote the set of lines through a point p with slope in the interval I_i, and define

$$\mathcal{R}_{i,j} = \{\ell_i \cap \ell_j : \ell_i \in \mathcal{L}((-1, 0), I_i), \ell_j \in \mathcal{L}((1, 0), I_j)\}.$$

Note that any point p in

$$\mathcal{R}_{i,i} = \{\ell_1 \cap \ell_2 : \ell_1 \in \mathcal{L}((-1, 0), I_i), \ell_2 \in \mathcal{L}((1, 0), I_i)\}$$

is joined to both $(-1, 0)$ and $(1, 0)$; however, p is also joined to $(0, 0)$ and so is not a solution to (1011). Therefore, each solution to (1011) must lie in one of the $n^2 - n$ regions $\mathcal{R}_{i,j}$, where $i \neq j$. Since the intervals I_i and I_j are disjoint, each region $\mathcal{R}_{i,j}$ is bounded by the four lines

$$\mathcal{L}((-1, 0), \{r_i\}), \mathcal{L}((-1, 0), \{s_i\}), \mathcal{L}((1, 0), \{r_j\}), \text{ and } \mathcal{L}((1, 0), \{s_j\}).$$

Further, $\mathcal{R}_{i,j}$ is either the interior of a bounded convex quadrilateral (when $0 \notin I_i \cup I_j$) or the interior of a bow-tie-shaped region (when $0 \in (I_i \cup I_j)$).

For each bounded region $\mathcal{R}_{i,j}$ with $i \neq j$, let $t_{i,j} \in \mathbb{Q}$ be an upper bound for the set of rationals

$$\{y - r_1 x, y - s_n x : (x, y) \in R_{i,j}\},$$

and choose $t > t_{i,j}$ for all i and j so that $t > 0$.

From above, (a, b) is a solution to (1011) and $(a, b) \in \mathcal{R}_{k,m}$. The line joining $(0, t)$ and (a, b) has slope $\frac{b-t}{a}$. If $a > 0$, then $t > b - r_1 a$ which implies $\frac{b-t}{a} < r_1$; and if $a < 0$, then $t > b - s_n a$ which implies $\frac{b-t}{a} > s_n$. Hence, we derive the contradiction that $\frac{b-t}{a} \notin \mathcal{U}$ and (a, b) cannot be a solution to the problem (1011). □

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Department of Mathematics and Computer Science, Mount Allison University, Sackville, NB, Canada, E4L 1E6
E-mail address: cbaker@mta.ca

Department of Mathematics, Ryerson University, Toronto, ON, Canada, M5B 2K3
E-mail address: abonato@ryerson.ca

Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada, B3H 3J5
E-mail address: nmckay@mathstat.dal.ca

Department of Mathematics and Statistics, Dalhousie University, Halifax NS, Canada, B3H 3J5
E-mail address: pralat@mathstat.dal.ca